The Combinatorial Inverse Eigenvalue Problem

Wayne Barrett, John Sinkovic, Curtis Nelson, Tianyi Yang, Nicole Malloy, William Sexton, Anne Lazenby, and Ryan Smith

October 12, 2012

Brigham Young University
Theorem

Given \( n \) real numbers \( \lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \ldots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n \) such that \( \mu_i \neq \mu_j \) for all \( i \neq j \), there exists an \( n \times n \) bordered matrix \( A = \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} b & M \end{bmatrix} \) with eigenvalues \( \lambda_1, \ldots, \lambda_n \), where \( M = \text{diag}(\mu_1, \mu_2, \ldots, \mu_{n-1}) \).

Moreover, there is an explicit formula for \( a \) and \( b \):

\[
a = \text{trace}(A) - \text{trace}(M) = \lambda_1 + \lambda_2 + \cdots + \lambda_n - \mu_1 - \cdots - \mu_{n-1}.
\]
Theorem

Given $2n - 1$ real numbers
Theorem

Given $2n - 1$ real numbers

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \ldots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n$$
Boley-Golub Theorem

Theorem

Given $2n - 1$ real numbers

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \ldots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n$$

such that $\mu_i \neq \mu_j$ for all $i \neq j$,
Given $2n - 1$ real numbers

$$
\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \ldots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n
$$

such that $\mu_i \neq \mu_j$ for all $i \neq j$,

there exists an $n \times n$ bordered matrix

$$
A = \begin{bmatrix}
a & b^T \\
b & M
\end{bmatrix}
$$
Boley-Golub Theorem

**Theorem**

Given $2n - 1$ real numbers

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \ldots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n$$

such that $\mu_i \neq \mu_j$ for all $i \neq j$,

there exists an $n \times n$ bordered matrix

$$A = \begin{bmatrix} a & b^T \\ b & M \end{bmatrix}$$

with eigenvalues $\lambda_1, \ldots, \lambda_n$. 

Theorem

Given $2n - 1$ real numbers

\[ \lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \ldots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n \]

such that $\mu_i \neq \mu_j$ for all $i \neq j$,

there exists an $n \times n$ bordered matrix

\[ A = \begin{bmatrix} a & b^T \\ b & M \end{bmatrix} \]

with eigenvalues $\lambda_1, \ldots, \lambda_n$, where $M = \text{diag}(\mu_1, \mu_2, \ldots, \mu_{n-1})$.
Theorem

Given $2n - 1$ real numbers

$$
\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \ldots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n
$$

such that $\mu_i \neq \mu_j$ for all $i \neq j$,

there exists an $n \times n$ bordered matrix

$$
A = \begin{bmatrix}
a & b^T \\
b & M
\end{bmatrix}
$$

with eigenvalues $\lambda_1, \ldots, \lambda_n$, where $M = \text{diag}(\mu_1, \mu_2, \ldots, \mu_{n-1})$

Moreover, there is an explicit formula for $a$ and $b^T = (b_1, b_2, \ldots, b_{n-1})$. 
Theorem

*Given* $2n - 1$ real numbers

$$
\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \ldots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n
$$

such that $\mu_i \neq \mu_j$ for all $i \neq j$,

there exists an $n \times n$ bordered matrix

$$
A = \begin{bmatrix}
a & b^T \\
b & M
\end{bmatrix}
$$

with eigenvalues $\lambda_1, \ldots, \lambda_n$, where $M = \text{diag}(\mu_1, \mu_2, \ldots, \mu_{n-1})$

Moreover, there is an explicit formula for $a$ and $b^T = (b_1, b_2, \ldots, b_{n-1})$.

$$
a = \text{trace } A - \text{trace } M
$$
Theorem

Given 2n − 1 real numbers

\[ \lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \ldots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n \]

such that \( \mu_i \neq \mu_j \) for all \( i \neq j \),

there exists an \( n \times n \) bordered matrix

\[
A = \begin{bmatrix} a & b^T \\ b & M \end{bmatrix}
\]

with eigenvalues \( \lambda_1, \ldots, \lambda_n \), where \( M = \text{diag}(\mu_1, \mu_2, \ldots, \mu_{n-1}) \)

Moreover, there is an explicit formula for \( a \) and \( b^T = (b_1, b_2, \ldots, b_{n-1}) \).

\[ a = \text{trace } A - \text{trace } M = \lambda_1 + \lambda_2 + \cdots + \lambda_n - \mu_1 - \cdots - \mu_{n-1}. \]
Boley-Golub Theorem

\[ b_i^2 = -\frac{\prod_{j=1}^{n} (\mu_i - \lambda_j)}{\left|\prod_{j=1}^{n-1} (\mu_i - \mu_j)\right|} \quad \text{for } 1 \leq i \leq n - 1. \]
Boley-Golub Theorem

\[ b_i^2 = -\frac{n}{\prod_{j=1}^{n-1} (\mu_i - \lambda_j)} \prod_{j=1, j\neq i}^{n-1} (\mu_i - \mu_j) \] for \( 1 \leq i \leq n - 1 \).

**Example:** $3 > 2 > 1 > 0 > -1 > -2 > -3$
Boley-Golub Theorem

\[ b_i^2 = \prod_{j=1}^{n-1} \left( \mu_i - \lambda_j \right) - \prod_{j=1, j \neq i}^{n-1} (\mu_i - \mu_j) \quad \text{for} \ 1 \leq i \leq n - 1. \]

**Example:** \( 3 > 2 > 1 > 0 > -1 > -2 > -3 \)

\[ a = 0 \quad b_1^2 = \frac{15}{8} \quad b_2^2 = \frac{9}{4} \quad b_3^2 = \frac{15}{8} \]
Boley-Golub Theorem

\[ b_i^2 = -\frac{\prod_{j=1}^{n} (\mu_i - \lambda_j)}{\prod_{j=1, j\neq i}^{n-1} (\mu_i - \mu_j)} \] for \( 1 \leq i \leq n - 1 \).

**Example:** \( 3 > 2 > 1 > 0 > -1 > -2 > -3 \)

\[ a = 0 \quad b_1^2 = \frac{15}{8} \quad b_2^2 = \frac{9}{4} \quad b_3^2 = \frac{15}{8} \]

\[ A = \begin{bmatrix}
0 & \sqrt{\frac{15}{8}} & 3/2 & \sqrt{\frac{15}{8}} \\
\sqrt{\frac{15}{8}} & 2 & 0 & 0 \\
3/2 & 0 & 0 & 0 \\
\sqrt{\frac{15}{8}} & 0 & 0 & -2
\end{bmatrix} \]
Boley-Golub Theorem

\[ b_i^2 = -\frac{\prod_{j=1}^{n-1}(\mu_i - \lambda_j)}{\prod_{j=1}^{n-1}(\mu_i - \mu_j)} \quad \text{for } 1 \leq i \leq n - 1. \]

Example: \(3 > 2 > 1 > 0 > -1 > -2 > -3\)

\[ a = 0 \quad b_1^2 = 15/8 \quad b_2^2 = 9/4 \quad b_3^2 = 15/8 \]

\[
A = \begin{bmatrix}
0 & \sqrt{15/8} & 3/2 & \sqrt{15/8} \\
\sqrt{15/8} & 2 & 0 & 0 \\
3/2 & 0 & 0 & 0 \\
\sqrt{15/8} & 0 & 0 & -2
\end{bmatrix}
\]

Evals: 3, 1, -1, -3
A common theme in inverse eigenvalue problems is to find a symmetric matrix $A$ with a particular zero/nonzero pattern such that $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A$ and $\mu_1, \ldots, \mu_{n-1}$ are the eigenvalues of $A$ with some row and corresponding column deleted.
A common theme in inverse eigenvalue problems is to find a symmetric matrix $A$ with a particular zero/nonzero pattern such that

- $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A$ and
- $\mu_1, \ldots, \mu_{n-1}$ are the eigenvalues of $A$ with some row and corresponding column deleted.
A common theme in inverse eigenvalue problems is to find a symmetric matrix $A$ with a particular zero/nonzero pattern such that

- $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A$ and
- $\mu_1, \ldots, \mu_{n-1}$ are the eigenvalues of $A$ with some row and corresponding column deleted.

A natural way to describe the zero/nonzero pattern is via an undirected graph.
Symmetric Matrix associated with a Graph

\[ S_n - \text{set of all } n \times n \text{ real symmetric matrices} \]
Symmetric Matrix associated with a Graph

\( S_n \) - set of all \( n \times n \) real symmetric matrices

Given \( A \in S_n \), let \( G(A) \) be the graph with

vertex set \( V = \{1, 2, \ldots, n\} \)
Symmetric Matrix associated with a Graph

$S_n$ - set of all $n \times n$ real symmetric matrices

Given $A \in S_n$, let $G(A)$ be the graph with

vertex set $V = \{1, 2, \ldots, n\}$ and

edge set $E = \{\{i, j\}|a_{ij} \neq 0\}$
Symmetric Matrix associated with a Graph

$S_n$ - set of all $n \times n$ real symmetric matrices

Given $A \in S_n$, let $G(A)$ be the graph with

- vertex set $V = \{1, 2, \ldots, n\}$ and
- edge set $E = \{\{i, j\}|a_{ij} \neq 0\}$

For any graph $G$, let $S(G) = \{A \in S_n \mid G(A) = G\}$
Symmetric Matrix associated with a Graph

\( S_n \) - set of all \( n \times n \) real symmetric matrices

Given \( A \in S_n \), let \( G(A) \) be the graph with

- vertex set \( V = \{1, 2, \ldots, n\} \) and
- edge set \( E = \{\{i, j\} | a_{ij} \neq 0\} \)

For any graph \( G \), let \( S(G) = \{A \in S_n \mid G(A) = G\} \)
Symmetric Matrix associated with a Graph

$S_n$ - set of all $n \times n$ real symmetric matrices

Given $A \in S_n$, let $G(A)$ be the graph with

- vertex set $V = \{1, 2, \ldots, n\}$ and
- edge set $E = \{\{i, j\} | a_{ij} \neq 0\}$

For any graph $G$, let $S(G) = \{A \in S_n | G(A) = G\}$

A matrix $A$ can be visualized as a graph where each non-zero entry $(i, j)$ corresponds to an edge between vertices $i$ and $j$. The matrix $A$ is an example of a matrix that corresponds to the graph $G$.

$$A = \begin{bmatrix} d_1 & a & b & 0 \\ a & d_2 & c & 0 \\ b & c & d_3 & d \\ 0 & 0 & d & d_4 \end{bmatrix} \in S(G)$$
Symmetric Matrix associated with a Graph

$S_n$ - set of all $n \times n$ real symmetric matrices

Given $A \in S_n$, let $G(A)$ be the graph with

- vertex set $V = \{1, 2, \ldots, n\}$ and
- edge set $E = \{\{i, j\}| a_{ij} \neq 0\}$

For any graph $G$, let $S(G) = \{A \in S_n \mid G(A) = G\}$

$$A = \begin{bmatrix} d_1 & a & b & 0 \\ a & d_2 & c & 0 \\ b & c & d_3 & d \\ 0 & 0 & d & d_4 \end{bmatrix} \in S(G)$$
Symmetric Matrix associated with a Graph

$S_n$ - set of all $n \times n$ real symmetric matrices

Given $A \in S_n$, let $G(A)$ be the graph with

vertex set $V = \{1, 2, \ldots, n\}$ and

element set $E = \{\{i,j\} | a_{ij} \neq 0\}$

For any graph $G$, let $S(G) = \{A \in S_n \mid G(A) = G\}$

$$A = \begin{bmatrix} d_1 & a & b & 0 \\ a & d_2 & c & 0 \\ b & c & d_3 & d \\ 0 & 0 & d & d_4 \end{bmatrix} \in S(G)$$
Example: $S_n = K_{1,n-1}$
Example: $S_n = K_{1,n-1}$

\[ A = \begin{bmatrix}
  d_1 & a_{12} & a_{13} & \cdots & a_{1n} \\
  a_{21} & d_2 & 0 & \cdots & 0 \\
  a_{31} & 0 & \ddots & \cdots & \vdots \\
  \vdots & \vdots & \ddots & \ddots & 0 \\
  a_{n1} & 0 & \cdots & 0 & d_n
\end{bmatrix} \in S(S_n) \]
Definition

A graph $T$ is a tree if

- $T$ is connected
- $T$ contains no cycle
Theorem (Duarte’s Theorem)

Let $T$ be a tree on $n$ vertices and let $v$ be a vertex of $T$. The theorem actually says more: If the degree of vertex $v$ is greater than 1, the eigenvalues $\mu_1, \ldots, \mu_{n-1}$ can be distributed in any way among the branches of $T - v$. In the last few months, Bryan Shader and his student Keivan Monfared generalized Duarte’s result to all connected graphs.
**Theorem (Duarte’s Theorem)**

Let $T$ be a tree on $n$ vertices and let $v$ be a vertex of $T$. Given $2n - 1$ distinct real numbers $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \cdots > \lambda_{n-1} > \mu_{n-1} > \lambda_n$, there exists a matrix $A \in S(T)$ such that $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A$ and $\mu_1, \ldots, \mu_{n-1}$ are the eigenvalues of $A(v)$.

The theorem actually says more: If the degree of vertex $v$ is greater than 1, the eigenvalues $\mu_1, \ldots, \mu_{n-1}$ can be distributed in any way among the branches of $T - v$. In the last few months, Bryan Shader and his student Keivan Monfared generalized Duarte’s result to all connected graphs.
Theorem (Duarte’s Theorem)

Let \( T \) be a tree on \( n \) vertices and let \( v \) be a vertex of \( T \). Given \( 2n - 1 \) distinct real numbers \( \lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \cdots > \lambda_{n-1} > \mu_{n-1} > \lambda_n \), there exists a matrix \( A \in S(T) \) such that
Theorem (Duarte’s Theorem)

Let $T$ be a tree on $n$ vertices and let $v$ be a vertex of $T$. Given $2n - 1$ distinct real numbers $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \cdots > \lambda_{n-1} > \mu_{n-1} > \lambda_n$, there exists a matrix $A \in S(T)$ such that

- $\lambda_1, \lambda_2, ..., \lambda_n$ are the eigenvalues of $A$ and
- $\mu_1, \ldots, \mu_{n-1}$ are the eigenvalues of $A(v)$.

The theorem actually says more: If the degree of vertex $v$ is greater than 1, the eigenvalues $\mu_1, ..., \mu_{n-1}$ can be distributed in any way among the branches of $T - v$.
Theorem (Duarte’s Theorem)

Let $T$ be a tree on $n$ vertices and let $v$ be a vertex of $T$. Given $2n - 1$ distinct real numbers $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \cdots > \lambda_{n-1} > \mu_{n-1} > \lambda_n$, there exists a matrix $A \in S(T)$ such that

- $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A$ and
- $\mu_1, \ldots, \mu_{n-1}$ are the eigenvalues of $A(v)$.

The theorem actually says more:
Theorem (Duarte’s Theorem)

Let $T$ be a tree on $n$ vertices and let $v$ be a vertex of $T$. Given $2n - 1$ distinct real numbers $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \cdots > \lambda_{n-1} > \mu_{n-1} > \lambda_n$, there exists a matrix $A \in S(T)$ such that

- $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A$ and
- $\mu_1, \ldots, \mu_{n-1}$ are the eigenvalues of $A(v)$.

The theorem actually says more:
If the degree of vertex $v$ is greater than 1, the eigenvalues $\mu_1, \ldots, \mu_{n-1}$ can be distributed in any way among the branches of $T - v$. 
Theorem (Duarte’s Theorem)

Let $T$ be a tree on $n$ vertices and let $v$ be a vertex of $T$. Given $2n-1$ distinct real numbers $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \cdots > \lambda_{n-1} > \mu_{n-1} > \lambda_n$, there exists a matrix $A \in S(T)$ such that

- $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A$ and
- $\mu_1, \ldots, \mu_{n-1}$ are the eigenvalues of $A(v)$.

The theorem actually says more:
If the degree of vertex $v$ is greater than 1, the eigenvalues $\mu_1, \ldots, \mu_{n-1}$ can be distributed in any way among the branches of $T - v$.

In the last few months, Bryan Shader and his student Keivan Monfared generalized Duarte’s result to all connected graphs.
The $\lambda$, $\mu$ Problem

**Question**

Given a graph $G$ on $n$ vertices, a vertex $v$ of $G$, is there an $A \in S(G)$ such that the $\lambda_i$’s are the eigenvalues of $A$ and the $\mu_i$’s are the eigenvalues of $A(v)$?

A complete answer cannot be given even for most trees, but can be given for complete graphs.
The $\lambda$, $\mu$ Problem

Question

Given a graph $G$ on $n$ vertices, a vertex $v$ of $G$, and numbers

\[ \lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \ldots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n, \]

is there an $A \in \mathcal{S}(G)$ such that the $\lambda_i$'s are the eigenvalues of $A$ and the $\mu_i$'s are the eigenvalues of $A(v)$?

A complete answer cannot be given even for most trees, but can be given for complete graphs.
The $\lambda$, $\mu$ Problem

**Question**

Given a graph $G$ on $n$ vertices, a vertex $v$ of $G$, and numbers

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \ldots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n,$$

is there an $A \in S(G)$ such that
The $\lambda$, $\mu$ Problem

Question

Given a graph $G$ on $n$ vertices, a vertex $v$ of $G$, and numbers

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \ldots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n,$$

is there an $A \in S(G)$ such that

- the $\lambda_i$'s are the eigenvalues of $A$
The $\lambda$, $\mu$ Problem

Question

Given a graph $G$ on $n$ vertices, a vertex $v$ of $G$, and numbers

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \ldots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n,$$

is there an $A \in S(G)$ such that

- the $\lambda_i$’s are the eigenvalues of $A$ and
- the $\mu_i$’s are the eigenvalues of $A(v)$?
The $\lambda$, $\mu$ Problem

**Question**

Given a graph $G$ on $n$ vertices, a vertex $v$ of $G$, and numbers

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \ldots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n,$$

is there an $A \in S(G)$ such that

- the $\lambda_i$’s are the eigenvalues of $A$ and
- the $\mu_i$’s are the eigenvalues of $A(v)$?

A complete answer cannot be given even for most trees,
The $\lambda$, $\mu$ Problem

**Question**

Given a graph $G$ on $n$ vertices, a vertex $v$ of $G$, and numbers

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \ldots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n,$$

is there an $A \in S(G)$ such that

- the $\lambda_i$’s are the eigenvalues of $A$ and
- the $\mu_i$’s are the eigenvalues of $A(v)$?

A complete answer cannot be given even for most trees, but can be given for complete graphs.
Complete Graphs

Definition

A graph is complete if every pair of vertices is adjacent.
Complete Graphs

Definition

A graph is complete if every pair of vertices is adjacent. A complete graph on $n$ vertices is denoted $K_n$. 

A graph is complete if every pair of vertices is adjacent. A complete graph on $n$ vertices is denoted $K_n$. 

\[ \begin{bmatrix} a & u & v & w \\ u & b & x & y \\ v & x & c & z \\ w & y & z & d \end{bmatrix} \in S(K_4) \text{ provided } uvwx \neq 0. \]
A graph is complete if every pair of vertices is adjacent. A complete graph on $n$ vertices is denoted $K_n$. 

$$
\begin{bmatrix}
    a & u & v & w \\
    u & b & x & y \\
    v & x & c & z \\
    w & y & z & d
\end{bmatrix} \in S(K_4) \text{ provided } uvwxyz \neq 0.
$$
Solution of the $\lambda$, $\mu$ Problem for $K_n$

Theorem

Given $2n - 1$ real numbers $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \ldots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n$, there exists $A \in S(K_n)$ such that $\lambda_i$'s are the eigenvalues of $A$ and $\mu_i$'s are the eigenvalues of $A(v)$, where $v$ is a vertex of $K_n$, if and only if $\mu_1 > \mu_{n-1}$ and the multiset $\{\mu_1, \mu_2, \ldots, \mu_{n-1}\} \not\subseteq \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$.

$n = 4$: $3 > 2 > 1 > 0 > -1 > -2 > -3$  
Yes $2 = 2 > 1 > 0 = 0 > -2 = -2$  
No $1 > 0 = 0 = 0 = 0 > -1$  
No $\text{Wayne Barrett (BYU)}$  
Combinatorial Inverse Eigenvalue Problem  
October 12, 2012 13 / 25
Solution of the $\lambda, \mu$ Problem for $K_n$

**Theorem**

Given $2n - 1$ real numbers

\[ \lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \ldots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n, \]
Theorem

Given $2n - 1$ real numbers

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \ldots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n,$$

there exists $A \in S(K_n)$ such that

- $\lambda_i$’s are the eigenvalues of $A$
Theorem

Given $2n - 1$ real numbers

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \ldots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n,$$

there exists $A \in S(K_n)$ such that

- $\lambda_i$’s are the eigenvalues of $A$ and
- $\mu_i$’s are the eigenvalues of $A(v)$, where $v$ is a vertex of $K_n$,

if and only if
Theorem

Given $2n - 1$ real numbers

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \ldots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n,$$

there exists $A \in S(K_n)$ such that

- $\lambda_i$’s are the eigenvalues of $A$ and
- $\mu_i$’s are the eigenvalues of $A(v)$, where $v$ is a vertex of $K_n$,

if and only if

$\mu_1 > \mu_{n-1}$
Solution of the $\lambda$, $\mu$ Problem for $K_n$

**Theorem**

*Given $2n - 1$ real numbers*

\[
\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \ldots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n,
\]

*there exists $A \in S(K_n)$ such that*

- $\lambda_i$’s are the eigenvalues of $A$ and
- $\mu_i$’s are the eigenvalues of $A(v)$, where $v$ is a vertex of $K_n$,

*if and only if*

$\mu_1 > \mu_{n-1}$ and the multiset $\{\mu_1, \mu_2, \ldots, \mu_{n-1}\} \not\subseteq \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. 
Solution of the $\lambda$, $\mu$ Problem for $K_n$

**Theorem**

Given $2n - 1$ real numbers

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \ldots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n,$$

there exists $A \in S(K_n)$ such that

- $\lambda_i$’s are the eigenvalues of $A$ and
- $\mu_i$’s are the eigenvalues of $A(v)$, where $v$ is a vertex of $K_n$.

if and only if

$\mu_1 > \mu_{n-1}$ and the multiset $\{\mu_1, \mu_2, \ldots, \mu_{n-1}\} \not\subseteq \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$.

$n = 4$: $3 > 2 > 1 > 0 > -1 > -2 > -3$
Solution of the $\lambda$, $\mu$ Problem for $K_n$

Theorem

Given $2n - 1$ real numbers

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \ldots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n,$$

there exists $A \in S(K_n)$ such that

- $\lambda_i$'s are the eigenvalues of $A$ and
- $\mu_i$'s are the eigenvalues of $A(v)$, where $v$ is a vertex of $K_n$,

if and only if

$\mu_1 > \mu_{n-1}$ and the multiset $\{\mu_1, \mu_2, \ldots, \mu_{n-1}\} \nsubseteq \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$.

$n = 4$: $3 > 2 > 1 > 0 > -1 > -2 > -3$ Yes
Solution of the $\lambda$, $\mu$ Problem for $K_n$

**Theorem**

Given $2n - 1$ real numbers

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \ldots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n,$$

there exists $A \in S(K_n)$ such that

- $\lambda_i$’s are the eigenvalues of $A$
- $\mu_i$’s are the eigenvalues of $A(v)$, where $v$ is a vertex of $K_n$,

if and only if

$\mu_1 > \mu_{n-1}$ and the multiset $\{\mu_1, \mu_2, \ldots, \mu_{n-1}\} \not\subseteq \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$.

$n = 4$:  

3 > 2 > 1 > 0 > -1 > -2 > -3  

Yes

2 = 2 > 1 > 0 = 0 > -2 = -2
Solution of the $\lambda$, $\mu$ Problem for $K_n$

**Theorem**

Given $2n - 1$ real numbers

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \ldots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n,$$

there exists $A \in S(K_n)$ such that

- $\lambda_i$’s are the eigenvalues of $A$ and
- $\mu_i$’s are the eigenvalues of $A(v)$, where $v$ is a vertex of $K_n$,

if and only if

$\mu_1 > \mu_{n-1}$ and the multiset $\{\mu_1, \mu_2, \ldots, \mu_{n-1}\} \nsubseteq \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$.

$n = 4$: $3 > 2 > 1 > 0 > -1 > -2 > -3$ \hspace{1cm} Yes

$2 = 2 > 1 > 0 = 0 > -2 = -2$ \hspace{1cm} No
Solution of the $\lambda, \mu$ Problem for $K_n$

**Theorem**

Given $2n - 1$ real numbers

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \ldots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n,$$

there exists $A \in S(K_n)$ such that

- $\lambda_i$’s are the eigenvalues of $A$ and
- $\mu_i$’s are the eigenvalues of $A(v)$, where $v$ is a vertex of $K_n$,

if and only if

$\mu_1 > \mu_{n-1}$ and the multiset \{\$\mu_1, \mu_2, \ldots, \mu_{n-1}\$\} \not\subset \{\$\lambda_1, \lambda_2, \ldots, \lambda_n\$\}.

$n = 4$:  

\[\begin{array}{ccccccc}
3 & > & 2 & > & 1 & > & 0 & > & -1 & > & -2 & > & -3 \\
2 & = & 2 & > & 1 & > & 0 & = & 0 & > & -2 & = & -2 \\
1 & > & 0 & = & 0 & = & 0 & = & 0 & = & 0 & > & -1
\end{array}\]

Yes

No
Solution of the $\lambda$, $\mu$ Problem for $K_n$

**Theorem**

*Given* $2n - 1$ real numbers

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \ldots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n,$$

*there exists* $A \in S(K_n)$ *such that*

- $\lambda_i$’s are the eigenvalues of $A$ and
- $\mu_i$’s are the eigenvalues of $A(v)$, where $v$ is a vertex of $K_n$,

*if and only if*

$\mu_1 > \mu_{n-1}$ *and the multiset* $\{\mu_1, \mu_2, \ldots, \mu_{n-1}\} \not\subseteq \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$.

$n = 4$: $3 > 2 > 1 > 0 > -1 > -2 > -3$ \hspace{1cm} Yes

$2 = 2 > 1 > 0 = 0 > -2 = -2$ \hspace{1cm} No

$1 > 0 = 0 = 0 = 0 = 0 > -1$ \hspace{1cm} No
First Necessary Condition

Observation

Assume $A \in S_n$ has eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $A(v)$ has eigenvalues $\mu_1 \geq \cdots \geq \mu_{n-1}$.
Observation

Assume $A \in S_n$ has eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $A(v)$ has eigenvalues $\mu_1 \geq \cdots \geq \mu_{n-1}$. If $\mu_1 = \mu_{n-1}$, then $A(v)$ is a scalar matrix.
Observation

Assume $A \in S_n$ has eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $A(v)$ has eigenvalues $\mu_1 \geq \cdots \geq \mu_{n-1}$. If $\mu_1 = \mu_{n-1}$, then $A(v)$ is a scalar matrix.

Proof: $A(v)$ is symmetric and all eigenvalues are equal.
First Necessary Condition

Observation

Assume $A \in S_n$ has eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $A(v)$ has eigenvalues $\mu_1 \geq \cdots \geq \mu_{n-1}$. If $\mu_1 = \mu_{n-1}$, then $A(v)$ is a scalar matrix.

Proof: $A(v)$ is symmetric and all eigenvalues are equal.

Corollary

If $A \in S(G)$ and $A(v)$ has eigenvalues $\mu_1 = \cdots = \mu_{n-1}$, then $G - v$ consists of isolated vertices.
Let $A \in S_n$ with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, and suppose that $B = A(1)$ has eigenvalues $\mu_1 \geq \cdots \geq \mu_{n-1}$.
Second Necessary Condition

Theorem

Let $A \in S_n$ with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, and suppose that $B = A(1)$ has eigenvalues $\mu_1 \geq \cdots \geq \mu_{n-1}$.

If the multiset $\{\mu_1, \mu_2, \ldots, \mu_{n-1}\} \subseteq \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$,
Theorem

Let $A \in S_n$ with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, and suppose that $B = A(1)$ has eigenvalues $\mu_1 \geq \cdots \geq \mu_{n-1}$.

If the multiset $\{\mu_1, \mu_2, \ldots, \mu_{n-1}\} \subseteq \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, then $A = a_{11} \oplus B$. 

Proof: By hypothesis, the eigenvalues of $A$ are $\lambda_k, \mu_1, \cdots, \mu_{n-1}$ for some $k$. So $a_{11} = \text{trace}(A) - \text{trace}(B) = \lambda_k$.

$2 \times 2$ minors: $E_2(A) - E_2(B) = n \sum_{i=2}^{n} (a_{11}a_{ii} - a_{i1}a_{1i}) = n \sum_{i=2}^{n} a_{11}a_{ii} - n \sum_{i=2}^{n} a_{i1}a_{1i} = \lambda_k \text{trace}(B) - n \sum_{i=2}^{n} a_{i1}a_{1i}$.
Let $A \in S_n$ with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, and suppose that $B = A(1)$ has eigenvalues $\mu_1 \geq \cdots \geq \mu_{n-1}$.

If the multiset $\{\mu_1, \mu_2, \ldots, \mu_{n-1}\} \subseteq \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, then $A = a_{11} \oplus B$.

Proof: By hypothesis, the eigenvalues of $A$ are $\lambda_k, \mu_1, \cdots, \mu_{n-1}$ for some $k$. 
Theorem

Let $A \in S_n$ with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, and suppose that $B = A(1)$ has eigenvalues $\mu_1 \geq \cdots \geq \mu_{n-1}$.

If the multiset $\{\mu_1, \mu_2, \ldots, \mu_{n-1}\} \subseteq \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, then $A = a_{11} \oplus B$.

Proof: By hypothesis, the eigenvalues of $A$ are $\lambda_k, \mu_1, \cdots, \mu_{n-1}$ for some $k$.

So $a_{11} = \text{trace } A - \text{trace } B = \lambda_k$. 
Second Necessary Condition

**Theorem**

Let $A \in S_n$ with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, and suppose that $B = A(1)$ has eigenvalues $\mu_1 \geq \cdots \geq \mu_{n-1}$.

If the multiset $\{\mu_1, \mu_2, \ldots, \mu_{n-1}\} \subseteq \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, then $A = a_{11} \oplus B$.

**Proof:** By hypothesis, the eigenvalues of $A$ are $\lambda_k, \mu_1, \cdots, \mu_{n-1}$ for some $k$.

So $a_{11} = \text{trace } A - \text{trace } B = \lambda_k$.

$2 \times 2$ minors:

$$E_2(A) - E_2(B)$$
Second Necessary Condition

Theorem

Let \( A \in S_n \) with eigenvalues \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \), and suppose that \( B = A(1) \) has eigenvalues \( \mu_1 \geq \cdots \geq \mu_{n-1} \).

If the multiset \( \{ \mu_1, \mu_2, \ldots, \mu_{n-1} \} \subseteq \{ \lambda_1, \lambda_2, \ldots, \lambda_n \} \), then \( A = a_{11} \oplus B \).

Proof: By hypothesis, the eigenvalues of \( A \) are \( \lambda_k, \mu_1, \cdots, \mu_{n-1} \) for some \( k \).

So \( a_{11} = \text{trace } A - \text{trace } B = \lambda_k \).

2 \times 2 minors:

\[
E_2(A) - E_2(B) = \sum_{i=2}^{n} (a_{11}a_{ii} - a_{i1}a_{1i})
\]
Theorem

Let $A \in S_n$ with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, and suppose that $B = A(1)$ has eigenvalues $\mu_1 \geq \cdots \geq \mu_{n-1}$.

If the multiset $\{\mu_1, \mu_2, \ldots, \mu_{n-1}\} \subseteq \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, then $A = a_{11} \oplus B$.

Proof: By hypothesis, the eigenvalues of $A$ are $\lambda_k, \mu_1, \cdots, \mu_{n-1}$ for some $k$.

So $a_{11} = \text{trace } A - \text{trace } B = \lambda_k$.

$2 \times 2$ minors:

$$E_2(A) - E_2(B) = \sum_{i=2}^{n} (a_{11}a_{ii} - a_{i1}a_{1i})$$

$$= \sum_{i=2}^{n} a_{11}a_{ii} - \sum_{i=2}^{n} a_{i1}a_{1i}$$
Theorem

Let \( A \in S_n \) with eigenvalues \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \), and suppose that \( B = A(1) \) has eigenvalues \( \mu_1 \geq \cdots \geq \mu_{n-1} \).

If the multiset \( \{ \mu_1, \mu_2, \ldots, \mu_{n-1} \} \subseteq \{ \lambda_1, \lambda_2, \ldots, \lambda_n \} \), then \( A = a_{11} \oplus B \).

Proof: By hypothesis, the eigenvalues of \( A \) are \( \lambda_k, \mu_1, \cdots, \mu_{n-1} \) for some \( k \).

So \( a_{11} = \text{trace } A - \text{trace } B = \lambda_k \).

\[ 2 \times 2 \text{ minors:} \]

\[ E_2(A) - E_2(B) = \sum_{i=2}^{n} (a_{11}a_{ii} - a_{i1}a_{1i}) \]

\[ = \sum_{i=2}^{n} a_{11}a_{ii} - \sum_{i=2}^{n} a_{i1}a_{1i} = \lambda_k \text{ trace } B - \sum_{i=2}^{n} a_{1i}^2 \]
Second Necessary Condition

\[ \lambda_k \text{trace } B - \sum_{i=2}^{n} a_{1i}^2 = E_2(A) - E_2(B) \]
Second Necessary Condition

\[ \lambda_k \text{ trace } B - \sum_{i=2}^{n} a_{1i}^2 = E_2(A) - E_2(B) = \sum_{1 \leq i < j \leq n} \lambda_i \lambda_j - \sum_{1 \leq i < j \leq n-1} \mu_i \mu_j \]
Second Necessary Condition

\[ \lambda_k \text{trace } B = \sum_{i=2}^{n} a_{1i}^2 = E_2(A) - E_2(B) = \sum_{1 \leq i < j \leq n} \lambda_i \lambda_j - \sum_{1 \leq i < j \leq n-1} \mu_i \mu_j \]

\[ = \sum_{i \neq k} \lambda_k \lambda_i \]
Second Necessary Condition

\[
\lambda_k \text{trace } B - \sum_{i=2}^{n} a_{1i}^2 = E_2(A) - E_2(B) = \sum_{1 \leq i < j \leq n} \lambda_i \lambda_j - \sum_{1 \leq i < j \leq n-1} \mu_i \mu_j
\]

\[
= \sum_{i \neq k} \lambda_k \lambda_i = \lambda_k \text{trace } B.
\]
Second Necessary Condition

\[ \lambda_k \text{trace } B - \sum_{i=2}^{n} a^2_{1i} = E_2(A) - E_2(B) = \sum_{1 \leq i < j \leq n} \lambda_i \lambda_j - \sum_{1 \leq i < j \leq n-1} \mu_i \mu_j \]

\[ = \sum_{i \neq k} \lambda_k \lambda_i = \lambda_k \text{trace } B. \]

Thus \( \sum_{i=2}^{n} a^2_{1i} = 0 \)
Second Necessary Condition

\[ \lambda_k \text{ trace } B - \sum_{i=2}^{n} a_{1i}^2 = E_2(A) - E_2(B) = \sum_{1 \leq i < j \leq n} \lambda_i \lambda_j - \sum_{1 \leq i < j \leq n-1} \mu_i \mu_j \]

\[ = \sum_{i \neq k} \lambda_k \lambda_i = \lambda_k \text{ trace } B. \]

Thus \[ \sum_{i=2}^{n} a_{1i}^2 = 0 \Rightarrow a_{1i} = 0 \text{ for all } i \]
\[ \lambda_k \text{ trace } B - \sum_{i=2}^{n} a_{1i}^2 = E_2(A) - E_2(B) = \sum_{1 \leq i < j \leq n} \lambda_i \lambda_j - \sum_{1 \leq i < j \leq n-1} \mu_i \mu_j \]

\[ = \sum_{i \neq k} \lambda_k \lambda_i = \lambda_k \text{ trace } B. \]

Thus \( \sum_{i=2}^{n} a_{1i}^2 = 0 \Rightarrow a_{1i} = 0 \) for all \( i \Rightarrow A = a_{11} \oplus B. \)
Second Necessary Condition

\[ \lambda_k \text{trace } B - \sum_{i=2}^{n} a_{1i}^2 = E_2(A) - E_2(B) = \sum_{1 \leq i < j \leq n} \lambda_i \lambda_j - \sum_{1 \leq i < j \leq n-1} \mu_i \mu_j \]

\[ = \sum_{i \neq k} \lambda_k \lambda_i = \lambda_k \text{trace } B. \]

Thus \[ \sum_{i=2}^{n} a_{1i}^2 = 0 \Rightarrow a_{1i} = 0 \text{ for all } i \Rightarrow A = a_{11} \oplus B. \]

Corollary

Assume \( A \in S(G) \) has eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_n \) and \( A(v) \) has eigenvalues \( \mu_1, \ldots, \mu_{n-1} \).
Second Necessary Condition

\[ \lambda_k \text{trace } B - \sum_{i=2}^{n} a_{1i}^2 = E_2(A) - E_2(B) = \sum_{1 \leq i < j \leq n} \lambda_i \lambda_j - \sum_{1 \leq i < j \leq n-1} \mu_i \mu_j \]

\[ = \sum_{i \neq k} \lambda_k \lambda_i = \lambda_k \text{trace } B. \]

Thus \[ \sum_{i=2}^{n} a_{1i}^2 = 0 \Rightarrow a_{1i} = 0 \text{ for all } i \Rightarrow A = a_{11} \oplus B. \]

Corollary

Assume \( A \in S(G) \) has eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_n \) and \( A(v) \) has eigenvalues \( \mu_1, \ldots, \mu_{n-1} \). If the multiset \( \{\mu_1, \mu_2, \ldots, \mu_{n-1}\} \subseteq \{\lambda_1, \lambda_2, \ldots, \lambda_n\}, \)
Second Necessary Condition

\[ \lambda_k \text{trace } B - \sum_{i=2}^{n} a_{1i}^2 = E_2(A) - E_2(B) = \sum_{1 \leq i < j \leq n} \lambda_i \lambda_j - \sum_{1 \leq i < j \leq n-1} \mu_i \mu_j \]

\[ = \sum_{i \neq k} \lambda_k \lambda_i = \lambda_k \text{trace } B. \]

Thus \( \sum_{i=2}^{n} a_{1i}^2 = 0 \Rightarrow a_{1i} = 0 \) for all \( i \Rightarrow A = a_{11} \oplus B. \)

**Corollary**

Assume \( A \in S(G) \) has eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_n \) and \( A(v) \) has eigenvalues \( \mu_1, \ldots, \mu_{n-1} \). If the multiset \( \{\mu_1, \mu_2, \ldots, \mu_{n-1}\} \subseteq \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \), then \( v \) is an isolated vertex of \( G \).
Sufficiency

Two of the key components
Sufficiency

Two of the key components

Theorem (Boley-Golub)

Given $2n - 1$ real numbers

$$
\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \ldots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n
$$

such that $\mu_i \neq \mu_j$ for all $i \neq j$, there exists an $n \times n$ bordered matrix

$$
A = \begin{bmatrix}
a & b^T \\
b & M
\end{bmatrix}
$$

with eigenvalues $\lambda_1, \ldots, \lambda_n$, where $M = \text{diag}(\mu_1, \mu_2, \ldots, \mu_{n-1})$
Sufficiency

Two of the key components

Theorem (Boley-Golub)

Given $2n - 1$ real numbers

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \ldots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n$$

such that $\mu_i \neq \mu_j$ for all $i \neq j$, there exists an $n \times n$ bordered matrix

$$A = \begin{bmatrix} a & b^T \\ b & M \end{bmatrix}$$

with eigenvalues $\lambda_1, \ldots, \lambda_n$, where $M = \text{diag}(\mu_1, \mu_2, \ldots, \mu_{n-1})$

Lemma

Let $n \geq 2$. Let $D$ be an $n \times n$ diagonal matrix whose diagonal entries are not all equal. Then there exists an $n \times n$ orthogonal matrix $Q$ such that $Q^T D Q \in S(K_n)$. 
Proof Outline for Sufficiency via an Example

\[ G = K_6 \]
Proof Outline for Sufficiency via an Example

\[ G = K_6 \]

\[ 3 \geq 2 \geq 2 \geq 1 \geq 1 \geq 1 \geq -1 \geq -2 \geq -2 \geq -2 \geq -3 \]
Proof Outline for Sufficiency via an Example

\[ G = K_6 \]

\[ 3 \geq 2 \geq 2 \geq 1 \geq 1 \geq 1 \geq -1 \geq -2 \geq -2 \geq -2 \geq -3 \]

taking out \( \{1, 1\} \):

\[ 3 \geq 2 \geq 2 \geq 1 \geq -1 \geq -2 \geq -2 \geq -2 \geq -3, \quad \{1, 1\} \]
Proof Outline for Sufficiency via an Example

\( G = K_6 \)

\[
3 \geq 2 \geq 2 \geq 1 > 1 \geq 1 \geq -1 \geq -2 \geq -2 \geq -2 \geq -3
\]

taking out \( \{1, 1\} \):

\[
3 \geq 2 \geq 2 \geq 1 \geq -1 \geq -2 \geq -2 \geq -2 \geq -3, \quad \{1, 1\}
\]

taking out \( \{-2, -2\} \):

\[
3 \geq 2 \geq 2 \geq 1 \geq -1 \geq -2 \geq -3, \quad \{1, 1\}, \quad \{-2, -2\}
\]
Proof Outline via an Example

\[ 3 \geq 2 \geq 2 \geq 1 \geq -1 \geq -2 \geq -3 \]
3 ≥ 2 ≥ 2 ≥ 1 ≥ −1 ≥ −2 ≥ −3

\[ A = \begin{bmatrix} a & b_1 & b_2 & b_3 \\ b_1 & 2 & 0 & 0 \\ b_2 & 0 & 1 & 0 \\ b_3 & 0 & 0 & -2 \end{bmatrix} \] by Boley-Golub
Proof Outline via an Example

\[ 3 \geq 2 \geq 2 \geq 1 \geq -1 \geq -2 \geq -3 \]

\[ A = \begin{bmatrix}
    a & b_1 & b_2 & b_3 \\
    b_1 & 2 & 0 & 0 \\
    b_2 & 0 & 1 & 0 \\
    b_3 & 0 & 0 & -2
\end{bmatrix} \text{ by Boley-Golub} \]

\[ \{2, 1, -2\} \not\subseteq \{3, 2, -1, -3\} \]
Proof Outline via an Example

\[ 3 \geq 2 \geq 2 \geq 1 \geq -1 \geq -2 \geq -3 \]

\[ A = \begin{bmatrix}
  a & b_1 & b_2 & b_3 \\
  b_1 & 2 & 0 & 0 \\
  b_2 & 0 & 1 & 0 \\
  b_3 & 0 & 0 & -2
\end{bmatrix} \text{ by Boley-Golub} \]

\[ \{2, 1, -2\} \not\subset \{3, 2, -1, -3\} \implies b_1, b_2, b_3 \text{ not all zero.} \]
Proof Outline via an Example

$3 \geq 2 \geq 2 \geq 1 \geq -1 \geq -2 \geq -3$

$$A = \begin{bmatrix} a & b_1 & b_2 & b_3 \\ b_1 & 2 & 0 & 0 \\ b_2 & 0 & 1 & 0 \\ b_3 & 0 & 0 & -2 \end{bmatrix}$$

by Boley-Golub

$$\{2, 1, -2\} \not\subset \{3, 2, -1, -3\} \implies b_1, b_2, b_3 \text{ not all zero.}$$

$$\{1, 1\}, \{−2, −2\}$$
Proof Outline via an Example

\[ 3 \geq 2 \geq 2 \geq 1 \geq -1 \geq -2 \geq -3 \]

\[
A = \begin{bmatrix}
  a & b_1 & b_2 & b_3 \\
  b_1 & 2 & 0 & 0 \\
  b_2 & 0 & 1 & 0 \\
  b_3 & 0 & 0 & -2
\end{bmatrix}
\]

by Boley-Golub

\{2, 1, -2\} \not\subseteq \{3, 2, -1, -3\} \implies b_1, b_2, b_3 \text{ not all zero.}

\{1, 1\}, \{-2, -2\}

\[
B = \begin{bmatrix}
  1 & 0 \\
  0 & -2
\end{bmatrix}
\]
Proof Outline via an Example

Direct sum of $A$ and $B$:

\[ A \oplus B = \begin{bmatrix}
    a & b_1 & b_2 & b_3 & 0 & 0 \\
    b_1 & 2 & 0 & 0 & 0 & 0 \\
    b_2 & 0 & 1 & 0 & 0 & 0 \\
    b_3 & 0 & 0 & -2 & 0 & 0 \\
    0 & 0 & 0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 0 & 0 & -2 \\
\end{bmatrix} \]
Proof Outline via an Example

Direct sum of $A$ and $B$:

$$A \oplus B = \begin{bmatrix} a & b_1 & b_2 & b_3 & 0 & 0 \\ b_1 & 2 & 0 & 0 & 0 & 0 \\ b_2 & 0 & 1 & 0 & 0 & 0 \\ b_3 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 \end{bmatrix}$$

$A \oplus B$ solves the $\lambda, \mu$ problem for

$$3 \geq 2 \geq 2 \geq 1 \geq 1 \geq 1 \geq -1 \geq -2 \geq -2 \geq -2 \geq -3.$$
Proof Outline via an Example

Direct sum of \( A \) and \( B \):

\[
A \oplus B = \begin{bmatrix}
  a & b_1 & b_2 & b_3 & 0 & 0 \\
  b_1 & 2 & 0 & 0 & 0 & 0 \\
  b_2 & 0 & 1 & 0 & 0 & 0 \\
  b_3 & 0 & 0 & -2 & 0 & 0 \\
  0 & 0 & 0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 0 & 0 & -2 \\
\end{bmatrix}
\]

\( A \oplus B \) solves the \( \lambda, \mu \) problem for

\[
3 \geq 2 \geq 2 \geq 1 \geq 1 \geq 1 \geq -1 \geq -2 \geq -2 \geq -2 \geq -3.
\]

Let \( D \) be the diagonal matrix obtained by deleting the first row and column of \( A \oplus B \)
Proof Outline via an Example

Let $D = \text{Diag}(2, 1, -2, 1, -2)$. 

By the Lemma, there exists an orthogonal matrix $Q$ of order 5 such that $E = Q^T D Q \in S(K_5)$.

Let $C = ([1] \oplus Q^T)(A \oplus B)([1] \oplus Q) = ([1] \oplus Q^T)\begin{bmatrix}
a & b \\
b_1 & b_2 & b_3 \\
b_2 & 0 & 1 & 0 & 0 & 0 \\
b_3 & 0 & 0 & -2 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & -2 \\
\end{bmatrix}([1] \oplus Q)$. 

$C$ solves the $\lambda, \mu$ problem for $3 \geq 2 \geq 2 \geq 1 \geq 1 \geq 1 \geq -1 \geq -2 \geq -2 \geq -3$.
Proof Outline via an Example

Let $D = \text{Diag} \ (2, 1, -2, 1, -2)$.

By the Lemma, there exists an orthogonal matrix $Q$ of order 5 such that $E = Q^T D Q \in S(K_5)$.
Let $D = \text{Diag} \ (2, 1, -2, 1, -2)$.

By the Lemma, there exists an orthogonal matrix $Q$ of order 5 such that $E = Q^T D Q \in S(K_5)$.

Let $C = ([1] \oplus Q^T)(A \oplus B)([1] \oplus Q)$.
Proof Outline via an Example

Let $D = \text{Diag}(2, 1, -2, 1, -2)$.

By the Lemma, there exists an orthogonal matrix $Q$ of order 5 such that $E = Q^T D Q \in S(K_5)$.

Let $C = ([1] \oplus Q^T)(A \oplus B)([1] \oplus Q)$

$$
= (1 \oplus Q^T) \begin{bmatrix}
  a & b_1 & b_2 & b_3 & 0 & 0 \\
  b_1 & 2 & 0 & 0 & 0 & 0 \\
  b_2 & 0 & 1 & 0 & 0 & 0 \\
  b_3 & 0 & 0 & -2 & 0 & 0 \\
  0 & 0 & 0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 0 & 0 & -2
\end{bmatrix} ([1] \oplus Q)
$$
Proof Outline via an Example

Let $D = \text{Diag} \ (2, 1, -2, 1, -2)$.

By the Lemma, there exists an orthogonal matrix $Q$ of order 5 such that $E = Q^T D Q \in S(K_5)$.

Let $C = ([1] \oplus Q^T)(A \oplus B)([1] \oplus Q)$

$$C = (1 \oplus Q^T) \begin{bmatrix}
 a & b_1 & b_2 & b_3 & 0 & 0 \\
 b_1 & 2 & 0 & 0 & 0 & 0 \\
 b_2 & 0 & 1 & 0 & 0 & 0 \\
 b_3 & 0 & 0 & -2 & 0 & 0 \\
 0 & 0 & 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 0 & 0 & -2 \\
\end{bmatrix} ([1] \oplus Q) = \begin{bmatrix}
 a & b^T \\
 b & E \\
\end{bmatrix}.$$
Proof Outline via an Example

Let $D = \text{Diag} (2, 1, -2, 1, -2)$.

By the Lemma, there exists an orthogonal matrix $Q$ of order 5 such that $E = Q^T D Q \in S(K_5)$.

Let $C = ([1] \oplus Q^T)(A \oplus B)([1] \oplus Q)$

$$C = (1 \oplus Q^T)
\begin{bmatrix}
 a & b_1 & b_2 & b_3 & 0 & 0 \\
 b_1 & 2 & 0 & 0 & 0 & 0 \\
 b_2 & 0 & 1 & 0 & 0 & 0 \\
 b_3 & 0 & 0 & -2 & 0 & 0 \\
 0 & 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 0 & -2 \\
\end{bmatrix}
\oplus Q
= ([1] \oplus Q)
\begin{bmatrix}
 a & b^T \\
 b & E \\
\end{bmatrix}.$$

$C$ solves the $\lambda, \mu$ problem for

$$3 \geq 2 \geq 2 \geq 1 \geq 1 \geq 1 \geq -1 \geq -2 \geq -2 \geq -2 \geq -3.$$
Proof Outline via an Example

Let $D = \text{Diag}(2, 1, -2, 1, -2)$.

By the Lemma, there exists an orthogonal matrix $Q$ of order 5 such that $E = Q^T D Q \in S(K_5)$.

Let $C = ([1] \oplus Q^T)(A \oplus B)([1] \oplus Q)$

$$= (1 \oplus Q^T) \begin{bmatrix} a & b_1 & b_2 & b_3 & 0 & 0 \\ b_1 & 2 & 0 & 0 & 0 & 0 \\ b_2 & 0 & 1 & 0 & 0 & 0 \\ b_3 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 \end{bmatrix} ([1] \oplus Q) = \begin{bmatrix} a & b^T \\ b & E \end{bmatrix}.$$  

$C$ solves the $\lambda, \mu$ problem for

$$3 \geq 2 \geq 2 \geq 1 \geq 1 \geq 1 \geq -1 \geq -2 \geq -2 \geq -2 \geq -3.$$  

and $b \neq 0$. 
Final step: Let $Q_2 = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$
**Proof Outline via an Example**

**Final step:** Let $Q_2 = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$

Choose $\theta$ such that $K = (Q_2^T \oplus I_{n-2}) \begin{bmatrix} a \\ b \\ E \end{bmatrix} (Q_2 \oplus I_{n-2})$ has all non-zero off-diagonal entries.
Final step: Let $Q_2 = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$

Choose $\theta$ such that $K = (Q_2^T \oplus I_{n-2}) \begin{bmatrix} a \\ b \\ E \end{bmatrix} (Q_2 \oplus I_{n-2})$ has all non-zero off-diagonal entries.

It takes a few lines to verify that this can be done.
Proof Outline via an Example

**Final step:** Let \( Q_2 = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \)

Choose \( \theta \) such that \( K = (Q_2^T \oplus I_{n-2}) \begin{bmatrix} a & b^T \\ b & E \end{bmatrix} (Q_2 \oplus I_{n-2}) \) has all non-zero off-diagonal entries.

It takes a few lines to verify that this can be done.

Then \( K \in S(K_6) \) solves the \( \lambda, \mu \) problem for

\[
3 \geq 2 \geq 2 \geq 1 \geq 1 \geq 1 \geq -1 \geq -2 \geq -2 \geq -2 \geq -3.
\]
Q1. Let $G$ be any connected graph on $n$ vertices and let $v$ be any vertex of $G$. Given $2^n - 1$ distinct real numbers $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \ldots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n$, where at least one $\geq$ is an $=$, is there a matrix $A \in S(G)$ such that $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A$ and $\mu_1, \ldots, \mu_{n-1}$ are the eigenvalues of $A(v)$. Even for small graphs the answer to this question can be complicated.
Q1. Let $G$ be any connected graph on $n$ vertices and let $v$ be any vertex of $G$. Given $2n - 1$ distinct real numbers

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n,$$

even for small graphs the answer to this question can be complicated.
Q1. Let $G$ be any connected graph on $n$ vertices and let $v$ be any vertex of $G$. Given $2n - 1$ distinct real numbers

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n,$$

where at least one $\geq$ is an $=$,
Q1. Let $G$ be any connected graph on $n$ vertices and let $v$ be any vertex of $G$. Given $2n - 1$ distinct real numbers

$$
\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n,
$$

where at least one $\geq$ is an $=$, is there a matrix $A \in S(G)$ such that

- $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A$ and
- $\mu_1, \ldots, \mu_{n-1}$ are the eigenvalues of $A(v)$. 

Even for small graphs the answer to this question can be complicated.
Q1. Let $G$ be any connected graph on $n$ vertices and let $v$ be any vertex of $G$. Given $2n-1$ distinct real numbers

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n,$$

where at least one $\geq$ is an $=$, is there a matrix $A \in \mathcal{S}(G)$ such that

- $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A$ and
- $\mu_1, \ldots, \mu_{n-1}$ are the eigenvalues of $A(v)$.

Even for small graphs the answer to this question can be complicated.
Paw/pendant vertex theorem

**Theorem**

Let $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \lambda_3 \geq \mu_3 \geq \lambda_4$. Let $G$ be the graph and let $v$ be the pendant vertex.

There exists $A \in \mathcal{S}(G)$ such that $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the eigenvalues of $A$ and $\mu_1, \mu_2, \mu_3$ are the eigenvalues of $A(v)$ if and only if all inequalities are strict. Exactly one of the inequalities is an equality and $\lambda_2 > \mu_2 > \lambda_3$.

One of the following holds:

- $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \lambda_3 = \mu_3 = \lambda_4$,
- $\lambda_1 > \mu_1 > \lambda_2 = \mu_2 = \lambda_3 > \mu_3 > \lambda_4$,
- $\lambda_1 > \mu_1 = \lambda_2 = \mu_2 > \lambda_3 > \mu_3 > \lambda_4$,
- $\lambda_1 = \mu_1 = \lambda_2 > \mu_2 > \lambda_3 > \mu_3 > \lambda_4$.

$\mu_2$ is the only equality and

$$\mu_2 = \lambda_3 \left( \frac{\lambda_1 - \mu_1}{\mu_1 - \lambda_2} \right) \left( \frac{\mu_1 - \lambda_2}{\mu_2 - \lambda_4} \right) + \left( \frac{\lambda_1 - \mu_1}{\mu_1 - \lambda_2} \right) \left( \frac{\mu_3 - \lambda_4}{\mu_2 - \lambda_3} \right) \left( \frac{\mu_3 - \lambda_4}{\mu_3 - \lambda_4} \right) \left( \frac{\lambda_1 - \mu_1}{\mu_1 - \lambda_2} \right) \left( \frac{\mu_3 - \lambda_4}{\mu_2 - \lambda_3} \right) + \left( \frac{\lambda_2 - \mu_3}{\mu_3 - \lambda_4} \right) \left( \frac{\mu_3 - \lambda_4}{\mu_3 - \lambda_4} \right) \left( \frac{\lambda_1 - \mu_1}{\mu_1 - \lambda_2} \right) \left( \frac{\mu_3 - \lambda_4}{\mu_2 - \lambda_3} \right) \left( \frac{\lambda_2 - \mu_3}{\mu_3 - \lambda_4} \right).$$

A similar condition if $\lambda_2 = \mu_2$ is the only equality.
Theorem

Let $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \lambda_3 \geq \mu_3 \geq \lambda_4$. Let $G$ be the graph and let $v$ be the pendant vertex. There exists $A \in S(G)$ such that $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the eigenvalues of $A$ and $\mu_1, \mu_2, \mu_3$ are the eigenvalues of $A(v)$ if and only if

One of the following holds:

1. $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \lambda_3 = \mu_3 = \lambda_4$,
2. $\lambda_1 > \mu_1 > \lambda_2 = \mu_2 = \lambda_3 > \mu_3 > \lambda_4$,
3. $\lambda_1 = \mu_1 = \lambda_2 > \mu_2 > \lambda_3 > \mu_3 > \lambda_4$.

$\mu_2$ is the only equality and $\mu_2 \neq \mu_1 \mu_3$. 

Wayne Barrett (BYU)
Theorem

Let \( \lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \lambda_3 \geq \mu_3 \geq \lambda_4 \). Let \( G \) be the graph and let \( v \) be the pendant vertex. There exists \( A \in S(G) \) such that \( \lambda_1, \lambda_2, \lambda_3, \lambda_4 \) are the eigenvalues of \( A \) and \( \mu_1, \mu_2, \mu_3 \) are the eigenvalues of \( A(v) \) if and only if

- all inequalities are strict.
Theorem

Let $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \lambda_3 \geq \mu_3 \geq \lambda_4$. Let $G$ be the graph and let $v$ be the pendant vertex. There exists $A \in S(G)$ such that $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the eigenvalues of $A$ and $\mu_1, \mu_2, \mu_3$ are the eigenvalues of $A(v)$ if and only if

- all inequalities are strict.
- exactly one of the inequalities is an equality and $\lambda_2 > \mu_2 > \lambda_3$. 

$\lambda_2 = \mu_2$ is the only equality and

$$\mu_2 = \lambda_3$$

and

$$\lambda_1 + \lambda_4 \neq \mu_1 + \mu_3.$$
Theorem

Let $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \lambda_3 \geq \mu_3 \geq \lambda_4$. Let $G$ be the graph \includegraphics[height=1cm]{graph.png} and let $v$ be the pendant vertex. There exists $A \in S(G)$ such that $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the eigenvalues of $A$ and $\mu_1, \mu_2, \mu_3$ are the eigenvalues of $A(v)$ if and only if

- all inequalities are strict.
- exactly one of the inequalities is an equality and $\lambda_2 > \mu_2 > \lambda_3$.
- One of the following holds: $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \lambda_3 = \mu_3 = \lambda_4$, $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 = \lambda_3 = \mu_3 > \lambda_4$, $\lambda_1 > \mu_1 = \lambda_2 = \mu_2 > \lambda_3 > \mu_3 > \lambda_4$, or $\lambda_1 = \mu_1 = \lambda_2 > \mu_2 > \lambda_3 > \mu_3 > \lambda_4$. 
Theorem

Let $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \lambda_3 \geq \mu_3 \geq \lambda_4$. Let $G$ be the graph and let $v$ be the pendant vertex. There exists $A \in S(G)$ such that $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the eigenvalues of $A$ and $\mu_1, \mu_2, \mu_3$ are the eigenvalues of $A(v)$ if and only if

- all inequalities are strict.
- exactly one of the inequalities is an equality and $\lambda_2 > \mu_2 > \lambda_3$.
- One of the following holds: $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \lambda_3 = \mu_3 = \lambda_4$, $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 = \lambda_3 = \mu_3 > \lambda_4$, $\lambda_1 > \mu_1 = \lambda_2 = \mu_2 > \lambda_3 > \mu_3 > \lambda_4$, or $\lambda_1 = \mu_1 = \lambda_2 > \mu_2 > \lambda_3 > \mu_3 > \lambda_4$.
- $\lambda_1 > \mu_1 > \lambda_2 = \mu_2 = \lambda_3 > \mu_3 > \lambda_4$ and $\lambda_1 + \lambda_4 \neq \mu_1 + \mu_3$. 
Theorem

Let $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \lambda_3 \geq \mu_3 \geq \lambda_4$. Let $G$ be the graph and let $v$ be the pendant vertex. There exists $A \in S(G)$ such that $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the eigenvalues of $A$ and $\mu_1, \mu_2, \mu_3$ are the eigenvalues of $A(v)$ if and only if

- all inequalities are strict.
- exactly one of the inequalities is an equality and $\lambda_2 > \mu_2 > \lambda_3$.
- One of the following holds: $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \lambda_3 = \mu_3 = \lambda_4$, $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 = \lambda_3 = \mu_3 > \lambda_4$, $\lambda_1 > \mu_1 = \lambda_2 = \mu_2 > \lambda_3 > \mu_3 > \lambda_4$, or $\lambda_1 = \mu_1 = \lambda_2 > \mu_2 > \lambda_3 > \mu_3 > \lambda_4$
- $\lambda_1 > \mu_1 > \lambda_2 = \mu_2 = \lambda_3 > \mu_3 > \lambda_4$ and $\lambda_1 + \lambda_4 \neq \mu_1 + \mu_3$.
- $\mu_2 = \lambda_3$ is the only equality.
Paw/pendant vertex theorem

**Theorem**

Let $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \lambda_3 \geq \mu_3 \geq \lambda_4$. Let $G$ be the graph and let $v$ be the pendant vertex. There exists $A \in S(G)$ such that $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the eigenvalues of $A$ and $\mu_1, \mu_2, \mu_3$ are the eigenvalues of $A(v)$ if and only if

- all inequalities are strict.
- exactly one of the inequalities is an equality and $\lambda_2 > \mu_2 > \lambda_3$.
- One of the following holds: $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \lambda_3 = \mu_3 = \lambda_4$, $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 = \lambda_3 = \mu_3 > \lambda_4$, $\lambda_1 > \mu_1 = \lambda_2 = \mu_2 > \lambda_3 > \mu_3 > \lambda_4$, or $\lambda_1 = \mu_1 = \lambda_2 > \mu_2 > \lambda_3 > \mu_3 > \lambda_4$.
- $\lambda_1 > \mu_1 > \lambda_2 = \mu_2 = \lambda_3 > \mu_3 > \lambda_4$ and $\lambda_1 + \lambda_4 \neq \mu_1 + \mu_3$.
- $\mu_2 = \lambda_3$ is the only equality and

$$\mu_2 \neq \frac{\mu_1 \mu_3 (\lambda_1 + \lambda_2 + \lambda_4 - \mu_1 - \mu_3) - \lambda_1 \lambda_2 \lambda_4}{(\lambda_1 - \mu_1)(\mu_1 - \lambda_2) + (\lambda_1 - \mu_1)(\mu_3 - \lambda_4) + (\lambda_2 - \mu_3)(\mu_3 - \lambda_4)}.$$
Theorem

Let $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \lambda_3 \geq \mu_3 \geq \lambda_4$. Let $G$ be the graph and let $v$ be the pendant vertex. There exists $A \in S(G)$ such that $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the eigenvalues of $A$ and $\mu_1, \mu_2, \mu_3$ are the eigenvalues of $A(v)$ if and only if

- all inequalities are strict.
- exactly one of the inequalities is an equality and $\lambda_2 > \mu_2 > \lambda_3$.
- One of the following holds: $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \lambda_3 = \mu_3 = \lambda_4$, $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 = \lambda_3 = \mu_3 > \lambda_4$, $\lambda_1 > \mu_1 = \lambda_2 = \mu_2 > \lambda_3 > \mu_3 > \lambda_4$, or $\lambda_1 = \mu_1 = \lambda_2 > \mu_2 > \lambda_3 > \mu_3 > \lambda_4$
- $\lambda_1 > \mu_1 > \lambda_2 = \mu_2 = \lambda_3 > \mu_3 > \lambda_4$ and $\lambda_1 + \lambda_4 \neq \mu_1 + \mu_3$.
- $\mu_2 = \lambda_3$ is the only equality and

$$
\mu_2 \neq \frac{\mu_1 \mu_3 (\lambda_1 + \lambda_2 + \lambda_4 - \mu_1 - \mu_3) - \lambda_1 \lambda_2 \lambda_4}{(\lambda_1 - \mu_1)(\mu_1 - \lambda_2) + (\lambda_1 - \mu_1)(\mu_3 - \lambda_4) + (\lambda_2 - \mu_3)(\mu_3 - \lambda_4)}.
$$

- a similar condition if $\lambda_2 = \mu_2$ is the only equality.
Q2. Let $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n$, let $G$ be any graph on $n$ vertices, let $v$ be any vertex of $G$, identify the edges $e$ in $G$ such that there exists a matrix $A \in S(G)$ such that the $\lambda_i$'s are the eigenvalues of $A$ and the $\mu_i$'s are the eigenvalues of $A(v)$.
Q2. Let $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n$, let $G$ be any graph on $n$ vertices, let $v$ be any vertex of $G$, identify the edges $e$ in $G$ such that

If there exists a matrix $A \in S(G)$ such that the $\lambda_i$'s are the eigenvalues of $A$ and the $\mu_i$'s are the eigenvalues of $A(v)$, there is a matrix $B \in S(G+e)$ such that the $\lambda_i$'s are the eigenvalues of $B$ and the $\mu_i'$s are the eigenvalues of $B(v)$?

John Sinkovic showed that this cannot always be done which I thought was surprising.
Q2. Let $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n$, let $G$ be any graph on $n$ vertices, let $v$ be any vertex of $G$,

Identify the edges $e$ in $G$ such that

If there exists a matrix $A \in S(G)$ such that the $\lambda_i$’s are the eigenvalues of $A$ and the $\mu_i$’s are the eigenvalues of $A(v)$,
Q2. Let $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n$, let $G$ be any graph on $n$ vertices, let $v$ be any vertex of $G$, Identify the edges $e$ in $G$ such that

If there exists a matrix $A \in S(G)$ such that the $\lambda_i$’s are the eigenvalues of $A$ and the $\mu_i$’s are the eigenvalues of $A(v)$,

there is a matrix $B \in S(G + e)$ such that the $\lambda_i$’s are the eigenvalues of $B$ and the $\mu_i$’s are the eigenvalues of $B(v)$?

John Sinkovic showed that this cannot always be done which I thought was surprising.
Open Questions

Q2. Let \( \lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n \), let \( G \) be any graph on \( n \) vertices, let \( v \) be any vertex of \( G \),

Identify the edges \( e \) in \( G \) such that

If there exists a matrix \( A \in \mathcal{S}(G) \) such that the \( \lambda_i \)'s are the eigenvalues of \( A \) and the \( \mu_i \)'s are the eigenvalues of \( A(v) \),

there is a matrix \( B \in \mathcal{S}(G + e) \) such that the \( \lambda_i \)'s are the eigenvalues of \( B \) and the \( \mu_i \)'s are the eigenvalues of \( B(v) \)?

John Sinkovic showed that this cannot always be done which I thought was surprising.