The Extended Combinatorial Inverse Eigenvalue Problem

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Theorem

Given $2n - 1$ real numbers $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n$ such that $\mu_i \neq \mu_j$ for all $i \neq j$,

there exists an $n \times n$ bordered matrix $A = \begin{bmatrix} a & b^T \\ b & M \end{bmatrix}$ with eigenvalues $\lambda_1, \ldots, \lambda_n$,

where $M = \text{diag}(\mu_1, \mu_2, \ldots, \mu_{n-1})$.

Moreover, there is an explicit formula for $a$ and $b^T = (b_1, b_2, \ldots, b_{n-1})$.

\[ a = \text{trace } A - \text{trace } M = \lambda_1 + \lambda_2 + \cdots + \lambda_n - \mu_1 - \cdots - \mu_{n-1}. \]
Theorem

Given $2n - 1$ real numbers
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\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \ldots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n
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$$A = \begin{bmatrix} a & b^T \\ b & M \end{bmatrix}$$
Boley-Golub Theorem

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Moreover, there is an explicit formula for $a$ and $b^T = (b_1, b_2, \ldots, b_{n-1})$.

$$a = \text{trace } A - \text{trace } M = \lambda_1 + \lambda_2 + \cdots + \lambda_n - \mu_1 - \cdots - \mu_{n-1}.$$
Boley-Golub Theorem

\[ b_i^2 = -\frac{\prod_{j=1}^{n} (\mu_i - \lambda_j)}{\prod_{j=1}^{n-1} (\mu_i - \mu_j)\prod_{j=1, j\neq i}^{n-1} (\mu_i - \mu_j)} \text{ for } 1 \leq i \leq n - 1. \]
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**Example:** \( 3 > 2 > 1 > 0 > -1 > -2 > -3 \)
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**Example:** \( 3 > 2 > 1 > 0 > -1 > -2 > -3 \)

\[ a = 0 \quad b_1^2 = 15/8 \quad b_2^2 = 9/4 \quad b_3^2 = 15/8 \]
Boley-Golub Theorem

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\[ A = \begin{bmatrix}
0 & \sqrt{\frac{15}{8}} & 3/2 & \sqrt{\frac{15}{8}} \\
\sqrt{\frac{15}{8}} & 2 & 0 & 0 \\
3/2 & 0 & 0 & 0 \\
\sqrt{\frac{15}{8}} & 0 & 0 & -2
\end{bmatrix} \]
Boley-Golub Theorem

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b_i^2 = -\frac{\prod_{j=1}^{n-1} (\mu_i - \lambda_j)}{\prod_{j=1}^{n-1} (\mu_i - \mu_j)} \quad \text{for } 1 \leq i \leq n - 1.
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Evals: 3, 1, -1, -3
A common theme in inverse eigenvalue problems is to find a symmetric matrix $A$ with a particular zero/nonzero pattern such that $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A$ and $\mu_1, \ldots, \mu_{n-1}$ are the eigenvalues of $A$ with some row and corresponding column deleted.
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A natural way to describe the zero/nonzero pattern is via an undirected graph.
Symmetric Matrix associated with a Graph

\[ S_n \] - set of all \( n \times n \) real symmetric matrices
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Given \( A \in S_n \), let \( G(A) \) be the graph with

vertex set \( V = \{1, 2, \ldots, n\} \)
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For any graph $G$, let $S(G) = \{A \in S_n \mid G(A) = G\}$
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![Graph Diagram]
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\[
A = \begin{bmatrix}
  d_1 & a & b & 0 \\
  a & d_2 & c & 0 \\
  b & c & d_3 & d \\
  0 & 0 & d & d_4 \\
\end{bmatrix} \in S(G)
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Example: $S_n = K_{1,n-1}$

\[ A = \begin{bmatrix}
  d_1 & a_{12} & a_{13} & \cdots & a_{1n} \\
  a_{21} & d_2 & 0 & \cdots & 0 \\
  a_{31} & 0 & d_3 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  a_{n1} & 0 & 0 & \cdots & d_n
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Example: \( S_n = K_{1,n-1} \)
Definition

A graph $T$ is a tree if
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- $T$ contains no cycle
A graph \( T \) is a tree if

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Theorem (Duarte’s Theorem)

Let $T$ be a tree on $n$ vertices and let $v$ be a vertex of $T$. Given $2n - 1$ distinct real numbers $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \cdots > \lambda_{n-1} > \mu_{n-1} > \lambda_n$, there exists a matrix $A \in S(T)$ such that $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A$ and $\mu_1, \ldots, \mu_{n-1}$ are the eigenvalues of $A(v)$. The theorem actually says more: If the degree of vertex $v$ is greater than 1, the eigenvalues $\mu_1, \ldots, \mu_{n-1}$ can be distributed in any way among the branches of $T - v$.

There are more results on the inverse eigenvalue problem for trees than other types of graphs.
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If the degree of vertex $v$ is greater than 1, the eigenvalues $\mu_1, \ldots, \mu_{n-1}$ can be distributed in any way among the branches of $T - v$. 
Inverse Eigenvalue Problem for Trees

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There are more results on the inverse eigenvalue problem for trees than other types of graphs.
The $\lambda, \mu$ Problem

**Question**

Given a graph $G$ on $n$ vertices

A complete answer cannot be given even for most trees, but can be given for complete graphs.
The $\lambda$, $\mu$ Problem

Question

Given a graph $G$ on $n$ vertices and numbers

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \ldots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n,$$

is there an $A \in \mathcal{S}(G)$ such that the $\lambda_i$'s are the eigenvalues of $A$ and the $\mu_i$'s are the eigenvalues of $A(v)$, where $v$ is a vertex of $G$?

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is there an $A \in \mathcal{S}(G)$ such that

- the $\lambda_i$’s are the eigenvalues of $A$
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A complete answer cannot be given even for most trees, but can be given for complete graphs.
Complete Graphs

Definition

A graph is complete if every pair of vertices is adjacent.
Complete Graphs

Definition

A graph is complete if every pair of vertices is adjacent. A complete graph on $n$ vertices is denoted $K_n$. 

A matrix $\begin{pmatrix} a & u & v & w \\ u & b & x & y \\ v & x & c & z \\ w & y & z & d \end{pmatrix} \in S(K_4)$ provided $uvwxyz \neq 0$. 

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**Complete Graphs**

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Solution of the $\lambda, \mu$ Problem for $K_n$

Given $2^n - 1$ real numbers $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n$, there exists $A \in S(K_n)$ such that $\lambda_i$'s are the eigenvalues of $A$ and $\mu_i$'s are the eigenvalues of $A(v)$, where $v$ is a vertex of $K_n$, if and only if $\mu_1 > \mu_{n-1}$ and the multiset \{\mu_1, \mu_2, \ldots, \mu_{n-1}\} \not\subseteq \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$.

$n = 4$:

Yes $2 = 2 > 1 > 0 > -1 > -2 > -3$

No $1 > 0 = 0 > -2 = -2$

No Way
Given 2n – 1 real numbers

\[ \lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \ldots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n, \]
Solution of the $\lambda$, $\mu$ Problem for $K_n$

**Theorem**

*Given $2n - 1$ real numbers

\[ \lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \ldots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n, \]

there exists $A \in S(K_n)$ such that

- $\lambda_i$’s are the eigenvalues of $A$*
Solution of the $\lambda$, $\mu$ Problem for $K_n$

Theorem

Given $2n - 1$ real numbers

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \ldots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n,$$

there exists $A \in S(K_n)$ such that

- $\lambda_i$'s are the eigenvalues of $A$ and
- $\mu_i$'s are the eigenvalues of $A(v)$, where $v$ is a vertex of $K_n$,

if and only if

$\mu_1 > \mu_{n-1}$ and the multiset $\{\mu_1, \mu_2, \ldots, \mu_{n-1}\}$ does not belong to $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$.
Solution of the $\lambda$, $\mu$ Problem for $K_n$

**Theorem**

Given $2n - 1$ real numbers

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \ldots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n,$$

there exists $A \in S(K_n)$ such that

- $\lambda_i$’s are the eigenvalues of $A$ and
- $\mu_i$’s are the eigenvalues of $A(v)$, where $v$ is a vertex of $K_n$.

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Solution of the $\lambda$, $\mu$ Problem for $K_n$

**Theorem**

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\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \ldots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n,
$$

*there exists $A \in S(K_n)$ such that*

- $\lambda_i$’s are the eigenvalues of $A$ and
- $\mu_i$’s are the eigenvalues of $A(v)$, where $v$ is a vertex of $K_n$,

*if and only if*

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\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \ldots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n,
\]

there exists \( A \in S(K_n) \) such that

- \( \lambda_i \)'s are the eigenvalues of \( A \) and
- \( \mu_i \)'s are the eigenvalues of \( A(\nu) \), where \( \nu \) is a vertex of \( K_n \),

if and only if

\( \mu_1 > \mu_{n-1} \) and the multiset \( \{ \mu_1, \mu_2, \ldots, \mu_{n-1} \} \nsubseteq \{ \lambda_1, \lambda_2, \ldots, \lambda_n \} \).

\( n = 4 \): \( 3 > 2 > 1 > 0 > -1 > -2 > -3 \)
Solution of the $\lambda$, $\mu$ Problem for $K_n$

**Theorem**

*Given 2n – 1 real numbers

\[ \lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \ldots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n, \]

there exists $A \in S(K_n)$ such that

- $\lambda_i$'s are the eigenvalues of $A$ and
- $\mu_i$'s are the eigenvalues of $A(v)$, where $v$ is a vertex of $K_n$,

if and only if

$\mu_1 > \mu_{n-1}$ and the multiset \{\mu_1, \mu_2, \ldots, \mu_{n-1}\} $\not\subseteq$ \{\lambda_1, \lambda_2, \ldots, \lambda_n\}.

$n = 4$: $3 > 2 > 1 > 0 > -1 > -2 > -3$  Yes
Solution of the $\lambda$, $\mu$ Problem for $K_n$

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Observation

Assume $A \in S_n$ has eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $A(v)$ has eigenvalues $\mu_1 \geq \cdots \geq \mu_{n-1}$.
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Proof: $A(v)$ is symmetric and all eigenvalues are equal.

Corollary

If $A \in S(G)$ and $A(v)$ has eigenvalues $\mu_1 = \cdots = \mu_{n-1}$, then $G-v$ consists of isolated vertices.
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Let $A \in S_n$ with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, and suppose that $B = A(1)$ has eigenvalues $\mu_1 \geq \cdots \geq \mu_{n-1}$. Then $A = a_{11} \oplus B$. 

Proof: By hypothesis, the eigenvalues of $A$ are $\lambda_k, \mu_1, \cdots, \mu_{n-1}$ for some $k$. So $a_{11} = \text{trace } A - \text{trace } B = \lambda_k$.

2 × 2 minors:

$$E_2(A) - E_2(B) = \sum_{i=2}^{n} (a_{11}a_{ii} - a_{i1}a_{1i}) = \lambda_k \text{trace } B - \sum_{i=2}^{n} a_{21}a_{1i}$$
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Second Necessary Condition

\[ \lambda_k \text{trace } B - \sum_{i=2}^{n} a_{1i}^2 = \text{E}_2(A) - \text{E}_2(B) \]
Second Necessary Condition

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\lambda_k \text{ trace } B - \sum_{i=2}^{n} a_{1i}^2 = E_2(A) - E_2(B) = \sum_{1 \leq i < j \leq n} \lambda_i \lambda_j - \sum_{1 \leq i < j \leq n-1} \mu_i \mu_j
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Thus \[ \sum_{i=2}^{n} a_{1i}^2 = 0 \Rightarrow a_{1i} = 0 \text{ for all } i \]
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Thus \( \sum_{i=2}^{n} a_{1i}^2 = 0 \Rightarrow a_{1i} = 0 \) for all \( i \Rightarrow A = a_11 \oplus B. \)
Second Necessary Condition

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Corollary

Assume \( A \in S(G) \) has eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_n \) and \( A(v) \) has eigenvalues \( \mu_1, \ldots, \mu_{n-1} \).
Second Necessary Condition

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Assume \( A \in S(G) \) has eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_n \) and \( A(\nu) \) has eigenvalues \( \mu_1, \ldots, \mu_{n-1} \). If the multiset \( \{\mu_1, \mu_2, \ldots, \mu_{n-1}\} \subseteq \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \),
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Sufficiency

Two of the key components
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Two of the key components

Theorem (Boley-Golub)

Given $2n - 1$ real numbers

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such that $\mu_i \neq \mu_j$ for all $i \neq j$, there exists an $n \times n$ bordered matrix

\[ A = \begin{bmatrix} a & b^T \\ b & M \end{bmatrix} \]

with eigenvalues $\lambda_1, \ldots, \lambda_n$, where $M = \text{diag}(\mu_1, \mu_2, \ldots, \mu_{n-1})$
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Lemma

Let $n \geq 2$. Let $D$ be an $n \times n$ diagonal matrix whose diagonal entries are not all equal. Then there exists an $n \times n$ orthogonal matrix $Q$ such that $Q^T D Q \in S(K_n)$. 
Proof Outline for Sufficiency via an Example

\[ G = K_6 \]
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\[ 3 \geq 2 \geq 2 \geq 1 \geq 1 \geq 1 \geq -1 \geq -2 \geq -2 \geq -2 \geq -3 \]
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taking out $\{1, 1\}$:

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\[
A = \begin{bmatrix}
a & b_1 & b_2 & b_3 \\
b_1 & 2 & 0 & 0 \\
b_2 & 0 & 1 & 0 \\
b_3 & 0 & 0 & -2 \\
\end{bmatrix}
\]

by Boley-Golub
3 ≥ 2 ≥ 2 ≥ 1 ≥ −1 ≥ −2 ≥ −3

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$\{2, 1, -2\} \not\subset \{3, 2, -1, -3\} \implies b_1, b_2, b_3$ not all zero.
Proof Outline via an Example

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\{1, 1\}, \{-2, -2\}
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\{1, 1\}, \{-2, -2\}

\[
B = \begin{bmatrix}
    1 & 0 \\
    0 & -2
\end{bmatrix}
\]
Direct sum of $A$ and $B$:

\[
A \oplus B = \begin{bmatrix}
a & b_1 & b_2 & b_3 & 0 & 0 \\
b_1 & 2 & 0 & 0 & 0 & 0 \\
b_2 & 0 & 1 & 0 & 0 & 0 \\
b_3 & 0 & 0 & -2 & 0 & 0 \\
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$A \oplus B$ solves the $\lambda, \mu$ problem for

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Let $D$ be the diagonal matrix obtained by deleting the first row and column of $A \oplus B$. 

Wayne Barrett (BYU)  
Inverse Eigenvalue Problem for Graphs  
July 12, 2012 20 / 25
Proof Outline via an Example

Let $D = \text{Diag} \ (2, 1, -2, 1, -2)$. 
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By the Lemma, there exists an orthogonal matrix $Q$ of order 5 such that $E = Q^T D Q \in S(K_5)$. 

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By the Lemma, there exists an orthogonal matrix $Q$ of order 5 such that $E = Q^T D Q \in S(K_5)$.

Let $C = ([1] \oplus Q^T)(A \oplus B)([1] \oplus Q)$

$$C = (1 \oplus Q^T) \begin{bmatrix} a & b_1 & b_2 & b_3 & 0 & 0 \\ b_1 & 2 & 0 & 0 & 0 & 0 \\ b_2 & 0 & 1 & 0 & 0 & 0 \\ b_3 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 \end{bmatrix} ([1] \oplus Q)$$
Proof Outline via an Example

Let \( D = \text{Diag} \left( 2, 1, -2, 1, -2 \right) \).

By the Lemma, there exists an orthogonal matrix \( Q \) of order 5 such that \( E = Q^T D Q \in S(K_5) \).

Let \( C = ([1] \oplus Q^T)(A \oplus B)([1] \oplus Q) \)

\[
\begin{bmatrix}
  a & b_1 & b_2 & b_3 & 0 & 0 \\
  b_1 & 2 & 0 & 0 & 0 & 0 \\
  b_2 & 0 & 1 & 0 & 0 & 0 \\
  b_3 & 0 & 0 & -2 & 0 & 0 \\
  0 & 0 & 0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 0 & 0 & -2 \\
\end{bmatrix}
\]

\( (1 \oplus Q^T)([1] \oplus Q) = \begin{bmatrix} a & b^T \\ b & E \end{bmatrix} \).
Let $D = \text{Diag}(2, 1, -2, 1, -2)$.

By the Lemma, there exists an orthogonal matrix $Q$ of order 5 such that $E = Q^T D Q \in S(K_5)$.

Let $C = ([1] \oplus Q^T)(A \oplus B)([1] \oplus Q)$

\[
\begin{bmatrix}
    a & b_1 & b_2 & b_3 & 0 & 0 \\
    b_1 & 2 & 0 & 0 & 0 & 0 \\
    b_2 & 0 & 1 & 0 & 0 & 0 \\
    b_3 & 0 & 0 & -2 & 0 & 0 \\
    0 & 0 & 0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 0 & 0 & -2
\end{bmatrix}
\]

$C$ solves the $\lambda, \mu$ problem for

\[3 \geq 2 \geq 2 \geq 1 \geq 1 \geq 1 \geq -1 \geq -2 \geq -2 \geq -2 \geq -3.\]
Proof Outline via an Example

Let \( D = \text{Diag} (2, 1, -2, 1, -2) \).

By the Lemma, there exists an orthogonal matrix \( Q \) of order 5 such that \( E = Q^T D Q \in S(K_5) \).

Let \( C = ([1] \oplus Q^T)(A \oplus B)([1] \oplus Q) \)

\[
= (1 \oplus Q^T) \begin{bmatrix}
    a & b_1 & b_2 & b_3 & 0 & 0 \\
    b_1 & 2 & 0 & 0 & 0 & 0 \\
    b_2 & 0 & 1 & 0 & 0 & 0 \\
    b_3 & 0 & 0 & -2 & 0 & 0 \\
    0 & 0 & 0 & 1 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & -2
\end{bmatrix}
\]

\(([1] \oplus Q) = \begin{bmatrix}
    a & b^T \\
    b & E
\end{bmatrix} \).

\( C \) solves the \( \lambda, \mu \) problem for

\[
3 \geq 2 \geq 2 \geq 1 \geq 1 \geq 1 \geq -1 \geq -2 \geq -2 \geq -2 \geq -3.
\]

and \( b \neq 0 \).
Final step: Let \( Q_2 = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \)
Final step: Let \( Q_2 = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \)

Choose \( \theta \) such that \( K = (Q_2^T \oplus I_{n-2}) \begin{bmatrix} a \\ b \\ E \end{bmatrix} (Q_2 \oplus I_{n-2}) \) has all non-zero off-diagonal entries.
Final step: Let \( Q_2 = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \)

Choose \( \theta \) such that \( K = (Q_2^T \oplus I_{n-2}) \begin{bmatrix} a \\ b \end{bmatrix} \begin{bmatrix} b^T \\ E \end{bmatrix} \) \((Q_2 \oplus I_{n-2})\) has all non-zero off-diagonal entries.

It seems reasonable that this can be done, but it takes some work to show it.
Final step: Let \( Q_2 = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \)

Choose \( \theta \) such that \( K = (Q_2^T \oplus I_{n-2}) \begin{bmatrix} a \\ b \\ E \end{bmatrix} (Q_2 \oplus I_{n-2}) \) has all non-zero off-diagonal entries.

It seems reasonable that this can be done, but it takes some work to show it.

Then \( K \in S(K_6) \) solves the \( \lambda, \mu \) problem for

\[ 3 \geq 2 \geq 2 \geq 1 \geq 1 \geq -1 \geq -2 \geq -2 \geq -2 \geq -3. \]
Q1. Let $G$ be any connected graph on $n$ vertices and let $v$ be any vertex of $G$. Given $2^n - 1$ distinct real numbers $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \cdots > \lambda_{n-1} > \mu_{n-1} > \lambda_n$, is there a matrix $A \in S(G)$ such that $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A$ and $\mu_1, \ldots, \mu_{n-1}$ are the eigenvalues of $A(v)$? It's true for all connected graphs on $n \leq 4$ vertices by construction.

Q2. Let $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n$, let $G$ be any graph on $n$ vertices, let $v$ be any vertex of $G$, and let $H$ be a graph obtained from $G$ by inserting one additional edge. If there exists a matrix $A \in S(G)$ such that the $\lambda_i$'s are the eigenvalues of $A$ and the $\mu_i$'s are the eigenvalues of $A(v)$, is there a matrix $B \in S(H)$ such that the $\lambda_i$'s are the eigenvalues of $B$ and the $\mu_i$'s are the eigenvalues of $B(v)$?
Open Questions

**Q1.** Let $G$ be any connected graph on $n$ vertices and let $v$ be any vertex of $G$. Given $2n-1$ distinct real numbers

$$\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \cdots > \lambda_{n-1} > \mu_{n-1} > \lambda_n,$$

is there a matrix $A \in S(G)$ such that $\lambda_i$'s are the eigenvalues of $A$ and $\mu_i$'s are the eigenvalues of $A(v)$?

It's true for all connected graphs on $n \leq 4$ vertices by construction.

**Q2.** Let $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n$, let $G$ be any graph on $n$ vertices, let $v$ be any vertex of $G$, and let $H$ be a graph obtained from $G$ by inserting one additional edge. If there exists a matrix $A \in S(G)$ such that the $\lambda_i$'s are the eigenvalues of $A$ and the $\mu_i$'s are the eigenvalues of $A(v)$, is there a matrix $B \in S(H)$ such that the $\lambda_i$'s are the eigenvalues of $B$ and the $\mu_i$'s are the eigenvalues of $B(v)$?
Open Questions

**Q1.** Let $G$ be any connected graph on $n$ vertices and let $v$ be any vertex of $G$. Given $2n - 1$ distinct real numbers

$$\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \cdots > \lambda_{n-1} > \mu_{n-1} > \lambda_n,$$

is there a matrix $A \in S(G)$ such that

- $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A$ and
- $\mu_1, \ldots, \mu_{n-1}$ are the eigenvalues of $A(v)$.
Open Questions

**Q1.** Let $G$ be any connected graph on $n$ vertices and let $v$ be any vertex of $G$. Given $2n - 1$ distinct real numbers

$$
\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \cdots > \lambda_{n-1} > \mu_{n-1} > \lambda_n,
$$
is there a matrix $A \in S(G)$ such that

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It’s true for all connected graphs on $n \leq 4$ vertices by construction.
Q1. Let $G$ be any connected graph on $n$ vertices and let $v$ be any vertex of $G$. Given $2n - 1$ distinct real numbers

$$\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \cdots > \lambda_{n-1} > \mu_{n-1} > \lambda_n,$$

is there a matrix $A \in S(G)$ such that

- $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A$ and
- $\mu_1, \ldots, \mu_{n-1}$ are the eigenvalues of $A(v)$.

It’s true for all connected graphs on $n \leq 4$ vertices by construction.

Q2. Let $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n$, let $G$ be any graph on $n$ vertices, let $v$ be any vertex of $G$,
Open Questions

Q1. Let $G$ be any connected graph on $n$ vertices and let $v$ be any vertex of $G$. Given $2n - 1$ distinct real numbers

$$
\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \cdots > \lambda_{n-1} > \mu_{n-1} > \lambda_n,
$$
is there a matrix $A \in S(G)$ such that

- $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A$ and
- $\mu_1, \ldots, \mu_{n-1}$ are the eigenvalues of $A(v)$.

It’s true for all connected graphs on $n \leq 4$ vertices by construction.

Q2. Let $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n$, let $G$ be any graph on $n$ vertices, let $v$ be any vertex of $G$, and let $H$ be a graph obtained from $G$ by inserting one additional edge.
**Open Questions**

**Q1.** Let $G$ be any connected graph on $n$ vertices and let $v$ be any vertex of $G$. Given $2n-1$ distinct real numbers

$$\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \cdots > \lambda_{n-1} > \mu_{n-1} > \lambda_n,$$

is there a matrix $A \in S(G)$ such that

- $\lambda_1, \lambda_2, ..., \lambda_n$ are the eigenvalues of $A$ and
- $\mu_1, \ldots, \mu_{n-1}$ are the eigenvalues of $A(v)$.

It’s true for all connected graphs on $n \leq 4$ vertices by construction.

**Q2.** Let $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n$, let $G$ be any graph on $n$ vertices, let $v$ be any vertex of $G$, and let $H$ be a graph obtained from $G$ by inserting one additional edge. If there exists a matrix $A \in S(G)$ such that the $\lambda_i$’s are the eigenvalues of $A$ and the $\mu_i$’s are the eigenvalues of $A(v)$,
Open Questions

**Q1.** Let $G$ be any connected graph on $n$ vertices and let $v$ be any vertex of $G$. Given $2n - 1$ distinct real numbers

$$\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \cdots > \lambda_{n-1} > \mu_{n-1} > \lambda_n,$$

is there a matrix $A \in S(G)$ such that

- $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A$ and
- $\mu_1, \ldots, \mu_{n-1}$ are the eigenvalues of $A(v)$.

It’s true for all connected graphs on $n \leq 4$ vertices by construction.

**Q2.** Let $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n$, let $G$ be any graph on $n$ vertices, let $v$ be any vertex of $G$, and let $H$ be a graph obtained from $G$ by inserting one additional edge.

If there exists a matrix $A \in S(G)$ such that the $\lambda_i$'s are the eigenvalues of $A$ and the $\mu_i$'s are the eigenvalues of $A(v)$,

is there a matrix $B \in S(H)$ such that the $\lambda_i$'s are the eigenvalues of $B$ and the $\mu'_i$'s are the eigenvalues of $B(v)$?
Q3. Given \( \lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n \), a graph \( G \) on \( n \) vertices, and a vertex \( v \) of \( G \), is there an \( A \in S(G) \) solving the \( \lambda, \mu \) problem?

Even for small graphs the answer to this question can be complicated.
Theorem

Let $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \lambda_3 \geq \mu_3 \geq \lambda_4$. Let $G$ be the graph and let $v$ be the pendant vertex.

There exists $A \in S(G)$ such that $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the eigenvalues of $A$ and $\mu_1, \mu_2, \mu_3$ are the eigenvalues of $A(v)$ if and only if all inequalities are strict.

Exactly one of the inequalities is an equality and $\lambda_2 > \mu_2 > \lambda_3$.

One of the following holds:

1. $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \lambda_3 = \mu_3 = \lambda_4$,
2. $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 = \lambda_3 = \mu_3 > \lambda_4$,
3. $\lambda_1 > \mu_1 = \lambda_2 = \mu_2 > \lambda_3 > \mu_3 > \lambda_4$,
4. $\lambda_1 = \mu_1 = \lambda_2 > \mu_2 > \lambda_3 > \mu_3 > \lambda_4$ and $\lambda_1 + \lambda_4 \neq \mu_1 + \mu_3$.

$\mu_2$ is the only equality and $\mu_2 \neq \mu_1 \mu_3$.

Wayne Barrett (BYU )
Inverse Eigenvalue Problem for Graphs
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Theorem

Let $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \lambda_3 \geq \mu_3 \geq \lambda_4$. Let $G$ be the graph and let $v$ be the pendant vertex. There exists $A \in S(G)$ such that $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the eigenvalues of $A$ and $\mu_1, \mu_2, \mu_3$ are the eigenvalues of $A(v)$ if and only if

1. All inequalities are strict.
2. Exactly one of the inequalities is an equality and $\lambda_2 > \mu_2 > \lambda_3$.
3. One of the following holds:
   - $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \lambda_3 = \mu_3 = \lambda_4$,
   - $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 = \lambda_3 = \mu_3 > \lambda_4$,
   - $\lambda_1 > \mu_1 = \lambda_2 = \mu_2 > \lambda_3 > \mu_3 > \lambda_4$,
   - $\lambda_1 = \mu_1 = \lambda_2 > \mu_2 > \lambda_3 > \mu_3 > \lambda_4$ and $\lambda_1 + \lambda_4 \neq \mu_1 + \mu_3$.
4. $\mu_2 = \lambda_3$ is the only equality and $\mu_2 \neq \mu_1 \mu_3$.

\[\lambda_1 - \lambda_2 \lambda_4 (\lambda_1 - \mu_1) (\mu_1 - \lambda_2) + (\lambda_1 - \mu_1)(\mu_3 - \lambda_4) + (\lambda_2 - \mu_3)(\mu_3 - \lambda_4) = 0.\]
Theorem

Let $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \lambda_3 \geq \mu_3 \geq \lambda_4$. Let $G$ be the graph and let $v$ be the pendant vertex. There exists $A \in S(G)$ such that $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the eigenvalues of $A$ and $\mu_1, \mu_2, \mu_3$ are the eigenvalues of $A(v)$ if and only if

- all inequalities are strict.
Paw/pendant vertex theorem

Theorem

Let $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \lambda_3 \geq \mu_3 \geq \lambda_4$. Let $G$ be the graph and let $v$ be the pendant vertex. There exists $A \in S(G)$ such that $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the eigenvalues of $A$ and $\mu_1, \mu_2, \mu_3$ are the eigenvalues of $A(v)$ if and only if

- all inequalities are strict.
- exactly one of the inequalities is an equality and $\lambda_2 > \mu_2 > \lambda_3$. 

\[ \text{Wayne Barrett (BYU )} \]
\[ \text{Inverse Eigenvalue Problem for Graphs} \]
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Theorem

Let $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \lambda_3 \geq \mu_3 \geq \lambda_4$. Let $G$ be the graph and let $v$ be the pendant vertex. There exists $A \in \mathcal{S}(G)$ such that $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the eigenvalues of $A$ and $\mu_1, \mu_2, \mu_3$ are the eigenvalues of $A(v)$ if and only if

- all inequalities are strict.
- exactly one of the inequalities is an equality and $\lambda_2 > \mu_2 > \lambda_3$.
- One of the following holds: $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \lambda_3 = \mu_3 = \lambda_4$, $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 = \lambda_3 = \mu_3 > \lambda_4$, $\lambda_1 > \mu_1 = \lambda_2 = \mu_2 > \lambda_3 > \mu_3 > \lambda_4$, or $\lambda_1 = \mu_1 = \lambda_2 > \mu_2 > \lambda_3 > \mu_3 > \lambda_4$.
Theorem

Let $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \lambda_3 \geq \mu_3 \geq \lambda_4$. Let $G$ be the graph and let $v$ be the pendant vertex. There exists $A \in S(G)$ such that $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the eigenvalues of $A$ and $\mu_1, \mu_2, \mu_3$ are the eigenvalues of $A(v)$ if and only if

- all inequalities are strict.
- exactly one of the inequalities is an equality and $\lambda_2 > \mu_2 > \lambda_3$.
- One of the following holds: $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \lambda_3 = \mu_3 = \lambda_4$, $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 = \lambda_3 = \mu_3 > \lambda_4$, $\lambda_1 > \mu_1 = \lambda_2 = \mu_2 > \lambda_3 > \mu_3 > \lambda_4$, or $\lambda_1 = \mu_1 = \lambda_2 > \mu_2 > \lambda_3 > \mu_3 > \lambda_4$.
- $\lambda_1 > \mu_1 > \lambda_2 = \mu_2 = \lambda_3 > \mu_3 > \lambda_4$ and $\lambda_1 + \lambda_4 \neq \mu_1 + \mu_3$. 
Theorem

Let $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \lambda_3 \geq \mu_3 \geq \lambda_4$. Let $G$ be the graph and let $v$ be the pendant vertex. There exists $A \in S(G)$ such that $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the eigenvalues of $A$ and $\mu_1, \mu_2, \mu_3$ are the eigenvalues of $A(v)$ if and only if

- all inequalities are strict.
- exactly one of the inequalities is an equality and $\lambda_2 > \mu_2 > \lambda_3$.
- One of the following holds: $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \lambda_3 = \mu_3 = \lambda_4$, $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 = \lambda_3 = \mu_3 > \lambda_4$, $\lambda_1 > \mu_1 = \lambda_2 = \mu_2 > \lambda_3 > \mu_3 > \lambda_4$, or $\lambda_1 = \mu_1 = \lambda_2 > \mu_2 > \lambda_3 > \mu_3 > \lambda_4$
- $\lambda_1 > \mu_1 > \lambda_2 = \mu_2 = \lambda_3 > \mu_3 > \lambda_4$ and $\lambda_1 + \lambda_4 \neq \mu_1 + \mu_3$.
- $\mu_2 = \lambda_3$ is the only equality.
Theorem

Let $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \lambda_3 \geq \mu_3 \geq \lambda_4$. Let $G$ be the graph and let $v$ be the pendant vertex. There exists $A \in S(G)$ such that $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the eigenvalues of $A$ and $\mu_1, \mu_2, \mu_3$ are the eigenvalues of $A(v)$ if and only if

- all inequalities are strict.
- exactly one of the inequalities is an equality and $\lambda_2 > \mu_2 > \lambda_3$.
- One of the following holds: $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \lambda_3 = \mu_3 = \lambda_4$, $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 = \lambda_3 = \mu_3 > \lambda_4$, $\lambda_1 > \mu_1 = \lambda_2 = \mu_2 > \lambda_3 > \mu_3 > \lambda_4$, or $\lambda_1 = \mu_1 = \lambda_2 > \mu_2 > \lambda_3 > \mu_3 > \lambda_4$.
- $\lambda_1 > \mu_1 > \lambda_2 = \mu_2 = \lambda_3 > \mu_3 > \lambda_4$ and $\lambda_1 + \lambda_4 \neq \mu_1 + \mu_3$.
- $\mu_2 = \lambda_3$ is the only equality and

$$\mu_2 \neq \frac{\mu_1 \mu_3 (\lambda_1 + \lambda_2 + \lambda_4 - \mu_1 - \mu_3) - \lambda_1 \lambda_2 \lambda_4}{(\lambda_1 - \mu_1)(\mu_1 - \lambda_2) + (\lambda_1 - \mu_1)(\mu_3 - \lambda_4) + (\lambda_2 - \mu_3)(\mu_3 - \lambda_4)}.$$
Theorem

Let $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \lambda_3 \geq \mu_3 \geq \lambda_4$. Let $G$ be the graph and let $v$ be the pendant vertex. There exists $A \in S(G)$ such that $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the eigenvalues of $A$ and $\mu_1, \mu_2, \mu_3$ are the eigenvalues of $A(v)$ if and only if

- all inequalities are strict.
- exactly one of the inequalities is an equality and $\lambda_2 > \mu_2 > \lambda_3$.
- One of the following holds: $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \lambda_3 = \mu_3 = \lambda_4$, $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 = \lambda_3 = \mu_3 > \lambda_4$, $\lambda_1 > \mu_1 = \lambda_2 = \mu_2 > \lambda_3 > \mu_3 > \lambda_4$, or $\lambda_1 = \mu_1 = \lambda_2 > \mu_2 > \lambda_3 > \mu_3 > \lambda_4$.
- $\lambda_1 > \mu_1 > \lambda_2 = \mu_2 = \lambda_3 > \mu_3 > \lambda_4$ and $\lambda_1 + \lambda_4 \neq \mu_1 + \mu_3$.
- $\mu_2 = \lambda_3$ is the only equality and

$$\mu_2 \neq \frac{\mu_1 \mu_3 (\lambda_1 + \lambda_2 + \lambda_4 - \mu_1 - \mu_3) - \lambda_1 \lambda_2 \lambda_4}{(\lambda_1 - \mu_1)(\mu_1 - \lambda_2) + (\lambda_1 - \mu_1)(\mu_3 - \lambda_4) + (\lambda_2 - \mu_3)(\mu_3 - \lambda_4)}.$$  

- a similar condition if $\lambda_2 = \mu_2$ is the only equality.