The Minimal Rank Problem and Forbidden Subgraphs

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Setup

\[ F \text{ - a field} \]
\[ G = (V, E) \text{ - a graph} \]
\[ V = \{1, 2, \ldots, n\} \]

\[ S(F, G) \]
- set of all symmetric \( n \times n \) matrices \( A \) with graph \( G \).

This means: For \( i \neq j \)

\[ a_{ij} \neq 0 \iff ij \in E \]

no condition on the diagonal entries
Example:

\[ S(F, \text{paw}) = \left\{ \begin{bmatrix} a & w & x & 0 \\ w & b & y & 0 \\ x & y & c & z \\ 0 & 0 & z & d \end{bmatrix} | a, b, c, d, w, x, y \in F, wxyz \neq 0 \right\} \]

The zeros correspond to the missing edges 14, 24.
Problem

Given a field $F$ and a graph $G$, find

$$mr(F,G) = \min \{ \text{rank } A | A \in S(F,G) \}$$

Idea of the Question: How much can you tell about a matrix if you only know where the zeros are?

Related to other topics:

- maximum multiplicity of eigenvalues
- degeneracies in chemical bonding theory
- Colin de Verdière graph parameter and planarity
Extreme examples

1. Complete graph $K_n$:

$$J_n = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & 1 \end{bmatrix} \in S(F, K_n) \implies \text{mr}(F, G) = 1, \quad n \geq 2$$

Fact: $\text{mr}(F, G) \leq 1 \iff G = K_m \cup K_{n-m}^c, \quad m \geq 2$
2. \( P_n \) \hspace{1cm} \text{Any } A \in S(F, P_n) \text{ has the form}

\[
A = \begin{bmatrix}
    a_1 & b_1 \\
    b_1 & a_2 & b_2 \\
    & b_2 & a_3 & \cdots \\
    \cdots & \cdots & b_{n-1} & b_n \\
    \end{bmatrix}, \hspace{1cm} b_i \neq 0
\]

Deleting the first column and last row gives an invertible lower triangular matrix

\[
\Rightarrow mr(F, P_n) \geq n - 1
\]
for $\text{char} F \neq 2$ (replace the 2's by zeros if $\text{char} F = 2$)

The rows sum to 0, so $L$ is singular $\implies \text{mr}(F, P_n) = n-1$
Focus of our work: For what graphs $G$ is $\text{mr}(F, G) \leq k$?

**Approach via forbidden subgraphs**

Observation: If $H$ is an induced subgraph of $G$, then any $B \in S(F, H)$ is a principal submatrix of an $A \in S(F, G)$

$$\implies \text{rank } B \leq \text{rank } A \implies \text{mr}(F, H) \leq \text{mr}(F, G)$$

**Example** $\text{mr}(F, P_{k+2}) = 4 + 2 - 1 = k + 1$

$$\implies P_{k+2} \text{ cannot be an induced subgraph of any graph } G \text{ with } \text{mr}(F, G) \leq k.$$
Definition. Fix a field $F$. The graph $S$ is a forbidden subgraph for the class of graphs $G_k = \{G \mid \text{mr}(F, G) \leq k\}$ if

1) $\text{mr}(F, S) = k + 1$

2) $\text{mr}(F, H) \leq k$ for any proper induced subgraph $H$ of $S$.

Let $\mathcal{F}_{k+1}$ be the set of forbidden subgraphs for $G_k$. Then

$$G \in G_k \iff \text{no graph in } \mathcal{F}_{k+1} \text{ is induced in } G.$$
\[ G \in \mathcal{G}_0 \iff G = K_n^c \quad \mathcal{F}_1 = \{K_2\} \]

\[ G \in \mathcal{G}_1 \iff G = K_m \cup K_{n-m}^c \quad \mathcal{F}_2 = \{P_3, 2K_2\} \]

Determining \( \mathcal{G}_2 \):

\( \mathcal{F}_3 \) depends on whether or not \( F \) is infinite and whether or not \( \text{char} \ F = 2 \).
Example:

Full House

clique sum of $K_4$ and $K_3$ on $K_2$

two missing edges: 15, 25
$F = \mathbb{R}$:

$$A = \begin{bmatrix}
1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
+ \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1
\end{bmatrix}
= \begin{bmatrix}
1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 0 \\
1 & 1 & 2 & 2 & 1 \\
0 & 0 & 1 & 1 & 1
\end{bmatrix}$$

$$A \in S(\mathbb{R}, \text{Fullhouse}) \quad \text{rank } A = 2$$

But if $\text{char } F = 2$, the 3,4 element is $1 + 1 = 0$, not 2 and this $A \not\in S(F, \text{Fullhouse})$
$F = F_2$:

Any $A \in S(F_2, \text{Fullhouse})$ has the form

\[
\begin{bmatrix}
  d_1 & 1 & 1 & 1 & 0 \\
  1 & d_2 & 1 & 1 & 0 \\
  1 & 1 & d_3 & 1 & 1 \\
  1 & 1 & 1 & d_4 & 1 \\
  0 & 0 & 1 & 1 & d_5 \\
\end{bmatrix}
\]

Know all off-diagonal entries now.

$A[145|235] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & d_5 \end{bmatrix}$ has determinant 1 so rank $A \geq 3$,

\[\implies \text{mr}(F_2, \text{Fullhouse}) \geq 3.\]
Theorem (Barrett, van der Holst, Loewy)

Infinite field Forbidden subgraphs

\[
\begin{cases}
\text{char } F \neq 2 & P_4, K_{3,3,3}, P_3 \cup K_2, 3K_2 \\
\text{char } F = 2 & P_4, K_{3,3,3}, P_3 \cup K_2, 3K_2
\end{cases}
\]

Finite field Forbidden subgraphs

\[
\begin{cases}
\text{char } F \neq 2 & \text{same 6 graphs as infinite case } + \text{ 3 more} \\
\text{char } F = 2 & \text{same 6 graphs as infinite case } + \text{ 4 more}
\end{cases}
\]
\[ F = F_2: \]

\[ \mathcal{F}_3 = \{ P_4, \quad \text{graph 1}, \quad \text{graph 2}, \quad \text{graph 3}, \quad P_3 \vee P_3, \quad P_3 \cup K_2, \quad 3K_2 \} \]

For each \( k \geq 3 \), a forbidden subgraph characterization of \( G_k \) exists, but is unknown for every field.

**Difficulties**

1) The number of forbidden subgraphs increases dramatically with \( k \).

2) It is very difficult to know if a given list is complete.
Incentive for studying finite fields

Theorem (Guoli Ding) If the field $F$ is finite, the set $\mathcal{F}_k$ of forbidden subgraphs is finite.

Work with Jason Grout and Don March:

Classification Theorem for $\mathcal{G}_k$ for any finite field of prime order

Enables us to generate such graphs but does not give an easy way to recognize them

In principle, it enables us to generate $\mathcal{F}_k$ automatically
Disconnected Graphs

Fact: if $G = \bigcup_{i=1}^{k} G_k$, $\text{mr}(F, G) = \sum_{i=1}^{k} \text{mr}(F, G_i)$

Example:

$\text{mr}(F, P_4 \cup K_2) = \text{mr}(F, P_4) + \text{mr}(F, K_2) = 3 + 1 = 4$.

Corollary: If $S \in \mathcal{F}_k$ is disconnected, then $S = \bigcup_{i=1}^{m} S_i$ with $S_i \in \mathcal{F}_{s_i}$ connected and $s_1 + s_2 + ... + s_m = k$
Example: disconnected graphs in $\mathcal{F}_4$ for an infinite field with char $F \neq 2$

$\mathcal{F}_3^c = \{P_4, \begin{tikzpicture} [scale=0.5] \draw[fill=white] (0,0) circle (0.1cm); \draw[fill=white] (1,1) circle (0.1cm); \draw[fill=white] (2,0) circle (0.1cm); \draw[fill=white] (3,1) circle (0.1cm); \draw (0,0) -- (1,1); \draw (1,1) -- (2,0); \draw (2,0) -- (3,1); \end{tikzpicture}, K_{3,3,3}\}$  $\mathcal{F}_2^c = \{P_3\}$  $\mathcal{F}_1 = \{K_2\}$

$3 + 1 : \quad P_4 \cup K_2, \quad \begin{tikzpicture} [scale=0.5] \draw[fill=white] (0,0) circle (0.1cm); \draw[fill=white] (1,1) circle (0.1cm); \draw[fill=white] (2,0) circle (0.1cm); \draw[fill=white] (3,1) circle (0.1cm); \draw (0,0) -- (1,1); \draw (1,1) -- (2,0); \draw (2,0) -- (3,1); \end{tikzpicture} \cup K_2, \quad \begin{tikzpicture} [scale=0.5] \draw[fill=white] (0,0) circle (0.1cm); \draw[fill=white] (1,1) circle (0.1cm); \draw[fill=white] (2,0) circle (0.1cm); \draw[fill=white] (3,1) circle (0.1cm); \draw (0,0) -- (1,1); \draw (1,1) -- (2,0); \draw (2,0) -- (3,1); \end{tikzpicture} \cup K_2, \quad K_{3,3,3} \cup K_2$

$2 + 2 : \quad P_3 \cup P_3 \quad 2 + 1 + 1 : \quad P_3 \cup 2K_2$

$1 + 1 + 1 + 1 : \quad 4K_2$
**AIM:** List of all graphs $G \in \mathcal{F}_4$ for any field $F$:

**FACTS:** Do not know if the list is finite if $F$ is infinite.

$P_5$ is the only graph on 5 vertices in $\mathcal{F}_4$.

**Next easiest case:** $G$ has a cut vertex

Definition: $G_1 \ast G_2$

$C_4$: $\bullet$, $K_{1,3}$ $\bullet$ $C_4 \ast K_{1,3}$

The dark vertex is necessarily a cut vertex.
Theorem. (L-Y Hsieh) (Barioli, Fallat, Hogben)
Let $F$ be a field and let $G = G_1 \ast G_2$ with cut vertex $v$. Then $\text{mr}(F,G)$ is the smaller of the two numbers

\[ \text{mr}(F,G_1) + \text{mr}(F,G_2), \quad \text{mr}(F,G_1 - v) + \text{mr}(F,G_2 - v) + 2 \]

Example: \[ \begin{array}{c}
\text{graph 1} = \text{graph 2} \ast K_2
\end{array} \]

$G_1 =$, $G_2 = K_2$, $\text{mr}(F, G_1) + \text{mr}(F, K_2) = 2 + 1 = 3$

$G_1 - v = P_3 \quad G_2 - v = K_1$

\[ \text{mr}(F, P_3) + \text{mr}(F, K_1) + 2 = 2 + 0 + 2 = 4 \]

$\implies \text{mr}(F, G) = 3$
**Application:** Let $H \in \mathcal{F}_k$. Then a vertex $v$ in $H$ is adjacent to at most two pendant vertices.

Proof: Suppose that $H \in \mathcal{F}_k$ has a vertex $v$ adjacent to $t \geq 2$ pendant vertices. Express $G = G_1 \ast K_{1,t}$.

mr($G_1$) + mr($K_{1,t}$) = mr($G_1$) + 2  since  $t \geq 2$

mr($G_1 - v$) + mr($K_{1,t} - v$) = mr($G_1 - v$) + mr($tK_1$) + 2
= mr($G_1 - v$) + 2

$\implies$ mr($G$) = mr($G_1 \ast K_{1,2}$) $\implies$ $G = G_1 \ast K_{1,2}$
$\implies$ $t = 2$
Graphs in $\mathcal{F}_4$ of the form $G = G_1 \ast K_{1,2}$

$\text{mr}(G) = 4 \implies \text{mr}(G_1 - v) + 2 = 4 \implies \text{mr}(G_1 - v) = 2$

$P_3$ is the unique graph in $\mathcal{F}_2$ so $G_1 - v$ contains $P_3$.

Use minimality of $G$ to deduce $G_1 - v = P_3$

$G$ has the form
Matches the computer generated list of graphs in $\mathcal{F}_4$ over $F_2$ containing a vertex adjacent to two pendant vertices.