Spanning 2-Forests, Resistance Distance, and the Laplacian Matrix

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Definition

A spanning subgraph of a graph $G$ is a subgraph with vertex set $V(G)$. A spanning tree is a spanning subgraph that is a tree.
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8 spanning trees.
Definition

A spanning 2-forest separating vertices $u$ and $v$ of a graph $G$ is a spanning forest with 2 components such that $u$ and $v$ are in distinct components.
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$u = 1, \, v = 2$

5 spanning 2-forests.
Electric circuits are modeled by graphs with a resistor on each edge.

Rules for combining resistances:

**Series Rule**

\[
\begin{align*}
\text{u} & \rightarrow r_1 \\
\text{v} & \rightarrow r_2 \\
\text{u} & \rightarrow r_1 + r_2 \\
\text{v} & \rightarrow \text{u} + \text{v}
\end{align*}
\]

We call \( r_1 + r_2 \) the effective resistance or resistance distance between \( u \) and \( v \).

**Parallel Rule**

\[
\begin{align*}
\text{x} & \rightarrow r_1 \\
\text{y} & \rightarrow r_2 \\
\text{x} & \rightarrow r_1 r_2 \\
\text{y} & \rightarrow \text{x} + \text{y}
\end{align*}
\]

Here \( r_1 r_2 \) is the effective resistance between \( x \) and \( y \).

Resistance distance is a metric on \( V(G) \).
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- **Series Rule**
  
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  \begin{align*}
  \text{Series Rule:} & \\
  \frac{1}{r_{1}} + \frac{1}{r_{2}} & = \frac{1}{r_{\text{effective}}}
  \end{align*}
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  Here \( r_{\text{effective}} \) is the effective resistance between \( u \) and \( v \).

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& \quad \Rightarrow \\
\end{align*}
\]

\[
\begin{align*}
& u \quad r_1 \quad r_2 \quad v \\
\end{align*}
\]

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\begin{align*}
& u \quad r_1 + r_2 \quad v \\
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**Parallel Rule**

\[ x \quad r_1 \quad r_2 \quad y \quad \rightarrow \quad x \quad \frac{r_1 r_2}{r_1 + r_2} \quad y \]
Electric circuits are modeled by graphs with a resistor on each edge.

Rules for combining resistances:

**Series Rule**

\[
\begin{align*}
  &u \xrightarrow{r_1} \middle\rightarrow\middle\rightarrow \xrightarrow{r_2} v \\
  &\quad \quad \xrightarrow{r_1 \parallel r_2} u \quad \quad \xrightarrow{v}
\end{align*}
\]

We call \( r_1 + r_2 \) the effective resistance or resistance distance between \( u \) and \( v \).

**Parallel Rule**

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\begin{align*}
  &x \xrightarrow{r_1} \xrightarrow{r_2} y \\
  &\quad \quad \xrightarrow{r_1 + r_2} x \quad \quad \xrightarrow{y}
\end{align*}
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Resistance distance is a metric on \( V(G) \).
For very simple circuits the series and parallel rules may suffice to calculate the resistance distance.

Assuming unit resistances on each edge, what is the effective resistance between nodes 1 and 2 in the circuit:

```
1
1
1
1
1
3
1
2
4
→
1
1
1
2
3
1
2
→
1
5
3
1
2
→
5
8
1
2
```

The effective resistance between nodes 1 and 2 is $\frac{5}{8}$. 
For very simple circuits the series and parallel rules may suffice to calculate the resistance distance.

Assuming unit resistances on each edge, what is the effective resistance between nodes 1 and 2 in the circuit:

\[
\begin{array}{c}
\text{1} \\
\text{2} \\
\text{3} \\
\text{4}
\end{array}
\]

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Assuming unit resistances on each edge, what is the effective resistance between nodes 1 and 2 in the circuit:

![Circuit Diagram]

It has to be less than 1.
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![Circuit Diagram]

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The effective resistance between nodes 1 and 2 is $\frac{5}{8}$. 
The number of spanning 2-forests separating 1 and 2 in $\begin{array}{c} 1 \\ \downarrow \\ 2 \\ \downarrow \\ 3 \\ \downarrow \\ 4 \end{array}$ is 5.
The number of spanning 2-forests separating 1 and 2 in $\mathcal{G}$ is 5.

The number of spanning trees of $\mathcal{G}$ is 8.
The number of spanning 2-forests separating 1 and 2 in $G$ is 5.

The number of spanning trees of $G$ is 8.

The effective resistance between nodes 1 and 2 in $G$ is $\frac{5}{8}$. 

\begin{itemize}
  \item \textbf{Theorem}
  \begin{align*}
    r_{G}(u,v) &= F_{G}(u,v) \frac{1}{T(G)} \\
  \end{align*}
\end{itemize}
The number of spanning 2-forests separating 1 and 2 in \( G \) is 5.

The number of spanning trees of \( G \) is 8.

The effective resistance between nodes 1 and 2 in \( G \) is \( \frac{5}{8} \).

**Theorem**

*Let \( G \) be an undirected graph, let \( u \) and \( v \) be vertices of \( G \),*
The number of spanning 2-forests separating 1 and 2 in $G$ is 5.

The number of spanning trees of $G$ is 8.

The effective resistance between nodes 1 and 2 in $G$ is $\frac{5}{8}$.

**Theorem**

Let $G$ be an undirected graph, let $u$ and $v$ be vertices of $G$, let $F_G(u, v)$ be the number of spanning 2-forests separating $u$ and $v$,
The number of spanning 2-forests separating 1 and 2 in $G$ is 5.

The number of spanning trees of $G$ is 8.

The effective resistance between nodes 1 and 2 in $G$ is $\frac{5}{8}$.

**Theorem**

Let $G$ be an undirected graph, let $u$ and $v$ be vertices of $G$, let $F_G(u, v)$ be the number of spanning 2-forests separating $u$ and $v$, let $T(G)$ be the number of spanning trees of $G$,
The number of spanning 2-forests separating 1 and 2 in \( G \) is 5.

The number of spanning trees of \( G \) is 8.

The effective resistance between nodes 1 and 2 in \( G \) is \( \frac{5}{8} \).

**Theorem**

Let \( G \) be an undirected graph, let \( u \) and \( v \) be vertices of \( G \), let \( \mathcal{F}_G(u, v) \) be the number of spanning 2-forests separating \( u \) and \( v \), let \( \mathcal{T}(G) \) be the number of spanning trees of \( G \), and let \( r_G(u, v) \) be the effective resistance between vertices \( u \) and \( v \).
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**Theorem**

Let $G$ be an undirected graph, let $u$ and $v$ be vertices of $G$, let $F_G(u, v)$ be the number of spanning 2-forests separating $u$ and $v$, let $T(G)$ be the number of spanning trees of $G$, and let $r_G(u, v)$ be the effective resistance between vertices $u$ and $v$. Then

$$r_G(u, v) = \frac{F_G(u, v)}{T(G)}.$$
Connection with the Laplacian matrix

Let $G$ be a graph on $n$ vertices and let $L(G)$ be its Laplacian matrix.

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Let $G$ be a graph on $n$ vertices and let $L(G)$ be its Laplacian matrix.

Then $\mathcal{T}(G) = \det L(j)$ where $L(j)$ is the matrix obtained by deleting the jth row and jth column. (Matrix Tree Theorem)
Connection with the Laplacian matrix

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$\mathcal{F}_G(i,j) = \det L(i,j)$, where $L(i,j)$ is the matrix obtained from $L$ by deleting the $i$th and $j$th rows and $i$th and $j$th columns.
Connection with the Laplacian matrix

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$\mathcal{F}_G(i, j) = \det L(i, j)$, where $L(i, j)$ is the matrix obtained from $L$ by deleting the $i$th and $j$th rows and $i$th and $j$th columns.

Then $r_G(i, j) = \frac{\det L(i, j)}{\det L(j)}$.

Alternatively, $r_G(i, j) = (e_i - e_j)^T L^\dagger (e_i - e_j)$,

where $L^\dagger$ is the Moore-Penrose inverse of $L$. 

Eigenvalues of $L(G)$: $0 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n$

Corresponding eigenvectors: $u_1, u_2, u_3, \cdots, u_n$
Connection with Spectral Graph Theory

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Corresponding eigenvectors: \( u_1, u_2, u_3, \cdots, u_n \)

Spectral decompositions: \( L = \lambda_2 u_2 u_2^T + \lambda_3 u_3 u_3^T + \cdots + \lambda_n u_n u_n^T \)
Connection with Spectral Graph Theory

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**Spectral decompositions:**

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L = \lambda_2 u_2 u_2^T + \lambda_3 u_3 u_3^T + \cdots + \lambda_n u_n u_n^T
\]

\[
L^\dagger = \frac{1}{\lambda_2} u_2 u_2^T + \frac{1}{\lambda_3} u_3 u_3^T + \cdots + \frac{1}{\lambda_n} u_n u_n^T
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$$r(i, j) = (e_i - e_j)^T L^\dagger (e_i - e_j)$$
Connection with Spectral Graph Theory

Eigenvalues of $L(G)$: $0 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n$

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\[ L^\dagger = \frac{1}{\lambda_2} u_2 u_2^T + \frac{1}{\lambda_3} u_3 u_3^T + \cdots + \frac{1}{\lambda_n} u_n u_n^T \]

\[ r(i, j) = (e_i - e_j)^T L^\dagger (e_i - e_j) = \frac{(u_{2i} - u_{2j})^2}{\lambda_2} + \frac{(u_{3i} - u_{3j})^2}{\lambda_3} + \cdots + \frac{(u_{ni} - u_{nj})^2}{\lambda_n} \]
Let $R$ be the $n \times n$ matrix whose $i, j$ entry is $r_G(i, j)$. 

Example $G$:
The Resistance Matrix

Let $R$ be the $n \times n$ matrix whose $i, j$ entry is $r_G(i, j)$.

Let $d = \text{diag}L^\dagger$ and let $e$ the all ones $n$–vector.
The Resistance Matrix

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$$R = de^T + ed^T - 2L^\dagger.$$
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Example

$G$:

```
1 -- 2 -- 5
|   |   |
3 -- 4
```
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Example

$G :$

![Graph Image]

$R = \frac{1}{21} \begin{bmatrix} 0 & 13 & 13 & 19 & 24 \\
13 & 0 & 10 & 12 & 19 \\
13 & 10 & 0 & 10 & 13 \\
19 & 12 & 10 & 0 & 13 \\
24 & 19 & 13 & 13 & 0 \end{bmatrix}$
The Resistance Matrix

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$G :$

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1 2 5
3 4
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```
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```

$$T(G) = 21$$
Cut vertices and 2-separators

Definition: A vertex \( w \) is a cut-vertex of the connected graph \( G \) if \( G - w \) is disconnected.

Definition: A pair of vertices \( i, j \) of a connected graph \( G \) is a 2-separator if \( G - \{i, j\} \) is disconnected.
Cut vertices and 2-separators

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Graphs with a cut vertex

Facts:
\[ T(G) = T(G_1) + T(G_2) \]
\[ r_G(u, v) = r_{G_1}(u, v) \text{ if } u, v \in G_1 \]
\[ r_G(u, v) = r_{G_1}(u, w) + r_{G_2}(v, w) \text{ if } u \in G_1, v \in G_2 \]

Are there reduction formulae for graphs with no cut vertex?
Yes, if the graph has a 2-separator.
Graphs with a cut vertex

Facts:
- $\mathcal{T}(G) = \mathcal{T}(G_1)\mathcal{T}(G_2)$
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Are there reduction formulae for graphs with no cut vertex?

Yes, if the graph has a 2-separator.
Graphs with a 2-separator

A 2-separation of a graph $G$ is a pair of subgraphs $G_1, G_2$ such that

- $V(G) = V(G_1) \cup V(G_2)$
- $|V(G_1) \cap V(G_2)| = 2$
- $E(G) = E(G_1) \cup E(G_2)$
- $E(G_1) \cap E(G_2) = \emptyset$
Graphs with a 2-separator

Decompose $G$ into 2 graphs:

$G_1$ and $G_2$
Graphs with a 2-separator

Decompose $G$ into 2 graphs:

$G$

$G_1$

$G_2$

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Proposition

Let $G$ be a graph with a 2-separation.
Proposition

Let $G$ be a graph with a 2-separation.

Then $\mathcal{T}(G) = \mathcal{T}(G_1)F_{G_2}(i,j) + \mathcal{T}(G_2)F_{G_1}(i,j)$.  

Example: Straight linear 2-tree (2-path) on $n$ vertices:

The graph $H$ has $n$ vertices. How many spanning trees does this graph have?
Spanning Tree Formula for graph with a 2-separation

Proposition

Let $G$ be a graph with a 2-separation. Then

$$\mathcal{T}(G) = \mathcal{T}(G_1)\mathcal{F}_{G_2}(i,j) + \mathcal{T}(G_2)\mathcal{F}_{G_1}(i,j).$$

Example: Straight linear 2-tree (2-path) on $n$ vertices:

$$H_n$$

1 3 5 \quad n-4 \quad n-2 \quad n

2 4 6 \quad n-3 \quad n-1
Spanning Tree Formula for graph with a 2-separation

Proposition

Let $G$ be a graph with a 2-separation.

Then $\mathcal{T}(G) = \mathcal{T}(G_1)\mathcal{F}_{G_2}(i,j) + \mathcal{T}(G_2)\mathcal{F}_{G_1}(i,j)$.

Example: Straight linear 2-tree (2-path) on $n$ vertices:

How many spanning trees does this graph have?
Example

\[ T(H_n) = F_{2n-2} \]

\[ F_0, F_1, F_2, F_3, F_4, F_5, F_6, F_7, F_8, \ldots \]

Apparently, \( T(H_n) = F_{2n-2} \) and \( F_0(H_n(n-1), n) = F_{2n-3} \).

Proceeding by induction, assume \( T(H_{n-1}) = F_{2n-4} \) and \( F_0(H_{n-1}(n-1), n) = F_{2n-5} \).
$\mathcal{T}(H_n)$

Example

$T(3) = 5$

$F_0 = 0$
$F_1 = 1$
$F_2 = 1$
$F_3 = 2$
$F_4 = 3$
$F_5 = 5$
$F_6 = 8$
$F_7 = 13$
$F_8 = 21$

$\ldots$

Fibonacci numbers

Apparently, $T(H_n) = F_{2n - 2}$ and $F_H(n, n - 1) = F_{2n - 3}$.

Proceeding by induction, assume $T(H_{n - 1}) = F_{2n - 4}$ and $F_H(n - 1, n - 1) = F_{2n - 5}$.
Example

\[ T(H_n) \]

\[ \mathcal{F}_{H_n}(n - 1, n) \]

\[ T(H_n) = 8 \]

\[ \mathcal{F}_{H_n}(n - 1, n) = 5 \]

\[ \mathcal{F}_{H_n}(n - 1, n) = 13 \]
Example

\[ T(H_n) \]
\[ \mathcal{F}_{H_n}(n - 1, n) \]

\[
\begin{array}{cccccccc}
F_0 & F_1 & F_2 & F_3 & F_4 & F_5 & F_6 & F_7 & F_8 & \cdots \\
0 & 1 & 1 & 2 & 3 & 5 & 8 & 13 & 21 & \cdots \\
\end{array}
\]

\( T(H_n) \) 8 21
\( \mathcal{F}_{H_n}(n - 1, n) \) 5 13

Fibonacci numbers
Example

\[ T(H_n) \]

\[ F_{H_n}(n - 1, n) \]

\[
\begin{array}{cccccccc}
F_0 & F_1 & F_2 & F_3 & F_4 & F_5 & F_6 & F_7 & F_8 & \cdots \\
0 & 1 & 1 & 2 & 3 & 5 & 8 & 13 & 21 & \cdots \\
\end{array}
\]

Fibonacci numbers

Apparently, \( T(H_n) = F_{2n-2} \)
Example

\[ T(H_n) \]

\[ F_{H_n}(n - 1, n) \]

\[ \begin{array}{cccccccc}
F_0 & F_1 & F_2 & F_3 & F_4 & F_5 & F_6 & F_7 & F_8 & \cdots \\
0 & 1 & 1 & 2 & 3 & 5 & 8 & 13 & 21 & \cdots \\
\end{array} \]

Fibonacci numbers

Apparently, \( T(H_n) = F_{2n-2} \) and \( F_{H_n}(n - 1, n) = F_{2n-3} \).
Example}

\[
\begin{array}{c}
\begin{array}{ccc}
\text{T} & \text{F}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{cccc}
\text{F}_0 & \text{F}_1 & \text{F}_2 & \text{F}_3 & \text{F}_4 & \text{F}_5 & \text{F}_6 & \text{F}_7 & \text{F}_8 & \cdots \\
0 & 1 & 1 & 2 & 3 & 5 & 8 & 13 & 21 & \cdots
\end{array}
\end{array}
\]

Fibonacci numbers

\[
\begin{align*}
\mathcal{T}(H_n) & \quad 8 \\
\mathcal{F}_H(n-1, n) & \quad 5
\end{align*}
\]

\[
\begin{align*}
\mathcal{T}(H_n) & = F_{2n-2} \\
\mathcal{F}_H(n-1, n) & = F_{2n-3}
\end{align*}
\]

Apparenty, proceeding by induction, assume \( \mathcal{T}(H_{n-1}) = F_{2n-4} \) and \( \mathcal{F}_{H_{n-1}}(n-1, n) = F_{2n-5} \).
Proof

\[ H_n \]

1 2 3 4 5 6 n-4 n-3 n-2 n

Two Separation

\[ T(H_n) = T(H_{n-1}) + F_{H_{n-1}}(n-4, n-1) + T(P_3) + F_{P_3}(n-2, n-1) = F_2(n-4) + F_2(n-4) + F_2(n-5) = F_2(n-4) + F_2(n-3) + F_2(n-2). \]
Proof

\[ H_n \]

Two Separation

\[ H_{n-1} \]

\[ P_3 \]
Proof

\[ T(H_n) = T(H_{n-1}) \mathcal{F}_{P_3}(n-2, n-1) + T(P_3) \mathcal{F}_{H_{n-1}}(n-2, n-1) \]
Proof

\[ \mathcal{T}(H_n) = \mathcal{T}(H_{n-1}) \mathcal{F}_{P_3}(n - 2, n - 1) + \mathcal{T}(P_3) \mathcal{F}_{H_{n-1}}(n - 2, n - 1) \]
\[ = F_{2n-4} \cdot 2 + 1 \cdot F_{2n-5} \]
Proof

\[ T(H_n) = T(H_{n-1}) \mathcal{F}_{P_3}(n-2, n-1) + T(P_3) \mathcal{F}_{H_{n-1}}(n-2, n-1) = F_{2n-4} \cdot 2 + 1 \cdot F_{2n-5} = F_{2n-4} + F_{2n-4} + F_{2n-5} \]
Proof

\[ \mathcal{T}(H_n) = \mathcal{T}(H_{n-1}) \mathcal{F}_{P_3}(n - 2, n - 1) + \mathcal{T}(P_3) \mathcal{F}_{H_{n-1}}(n - 2, n - 1) \]

\[ = F_{2n-4} \cdot 2 + 1 \cdot F_{2n-5} = F_{2n-4} + F_{2n-4} + F_{2n-5} \]

\[ = F_{2n-4} + F_{2n-3} \]
Proof

\[ \mathcal{T}(H_n) = \mathcal{T}(H_{n-1}) \mathcal{F}_{P_3}(n-2, n-1) + \mathcal{T}(P_3) \mathcal{F}_{H_{n-1}}(n-2, n-1) \]

\[ = F_{2n-4} \cdot 2 + 1 \cdot F_{2n-5} = F_{2n-4} + F_{2n-4} + F_{2n-5} \]

\[ = F_{2n-4} + F_{2n-3} = F_{2n-2}. \]
Theorem
Let $H$ be a graph with a 2-separator \{i, j\} and $G$ the graph obtained by performing a 2-switch on i and j. Then $T(G) = T(H)$. Moreover, if $u$ and $v$ are both in $H_1$ or both in $H_2$, then $F_G(u, v) = F_H(u, v)$ and $r_G(u, v) = r_H(u, v)$.
Theorem

Let $H$ be a graph with a 2-separator $\{i, j\}$ and $G$ the graph obtained by performing a 2-switch on $i$ and $j$. 
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Then $\mathcal{T}(G) = \mathcal{T}(H)$.
Theorem

Let $H$ be a graph with a 2-separator $\{i,j\}$ and $G$ the graph obtained by performing a 2-switch on $i$ and $j$.

- Then $T(G) = T(H)$.
- Moreover, if $u$ and $v$ are both in $H_1$ or both in $H_2$, then

$$F_G(u, v) = F_H(u, v) \quad \text{and} \quad r_G(u, v) = r_H(u, v).$$
Theorem

Then $T(H_n) = F_{2n} - 2$ and for any two vertices $u$ and $v$ $F_{H_n}(u, v) = F_{2n} - 1 + F_{v-u} + F_{2n-u-v+1} + F_{n-1} \left[ F_n + u - v - 2 (v-u-5) F_{v-u} + 2 (v-u+1) F_{v-u} \right]$, $L_j = F_{j-1} + F_{j+1}$, $r_{H_n}(u, v) = F_{H_n}(u, v) / F_{2n} - 2$.

This formula was obtained by repeatedly applying a circuit rule called the $\Delta Y$ transform.
Theorem

Then $T(H_n) = F_{2n-2}$
Then $\mathcal{T}(H_n) = F_{2n-2}$ and for any two vertices $u$ and $v$

$$\mathcal{F}_{H_n}(u, v) = F_{n-1}^2 + F_{v-u}^2 F_{n-u-v+1}^2 + \frac{F_{n-1}}{5} \left[ F_{n+u-v-2}((v-u)L_{v-u} - F_{v-u}) ight.$$ \n
$$+ F_{n+u-v-1}((v-u-5)F_{v-u+1} + 2(v-u+1)F_{v-u}) \right],$$
Theorem

Then $T(H_n) = F_{2n-2}$ and for any two vertices $u$ and $v$

$$F_{H_n}(u, v) = F_{n-1}^2 + F_{v-u}^2 F_{n-u-v+1}^2 + \frac{F_{n-1}}{5} [F_{n+u-v-2}((v-u)L_{v-u} - F_{v-u})$$
$$+ F_{n+u-v-1}((v-u-5)F_{v-u+1} + 2(v-u+1)F_{v-u})],$$

$L_j = F_{j-1} + F_{j+1}$,
Straight 2-path on $n$ vertices

Theorem

Then $T(H_n) = F_{2n-2}$ and for any two vertices $u$ and $v$

$$F_{H_n}(u, v) = F_{n-1}^2 + F_{v-u}^2 F_{n-u-v+1}^2 + \frac{F_{n-1}}{5} \left[ F_{n+u-v-2} ((v-u)L_{v-u} - F_{v-u}) ight]$$

$$+ F_{n+u-v-1} ((v-u-5)F_{v-u+1} + 2(v-u+1)F_{v-u})],$$

$$L_j = F_{j-1} + F_{j+1},$$

$$r_{H_n}(u, v) = F_{H_n}(u, v)/F_{2n-2}.$$
Straight 2-path on $n$ vertices

Theorem

Then $\mathcal{T}(H_n) = F_{2n-2}$ and for any two vertices $u$ and $v$

$$\mathcal{F}_{H_n}(u, v) = F_{n-1}^2 + F_{v-u}^2 F_{n-u-v+1}^2 + \frac{F_{n-1}}{5} [F_{n+u-v-2}((v-u)L_{v-u} - F_{v-u})$$

$$+ F_{n+u-v-1}((v-u-5)F_{v-u+1} + 2(v-u+1)F_{v-u})]$$

$$L_j = F_{j-1} + F_{j+1},$$

$$r_{H_n}(u, v) = \mathcal{F}_{H_n}(u, v)/F_{2n-2}.$$ 

This formula was obtained by repeatedly applying a circuit rule called the $\Delta Y$ transform.
Corollary

\[ F_{H_n}(1, n) = \frac{(n - 1)F_{2n-2} + 4F^2_{n-1}}{5}. \]
Corollary

\[ F_{H_n}(1, n) = \frac{(n - 1)F_{2n-2} + 4F_{n-1}^2}{5}. \]

Moreover, this is the maximum number of spanning 2-forests separating two vertices in the graph.
Corollary

\[ F_{H_n}(1, n) = \frac{(n - 1)F_{2n-2} + 4F^2_{n-1}}{5}. \]

Moreover, this is the maximum number of spanning 2-forests separating two vertices in the graph.

Dividing by \(F_{2n-2}\) gives the resistance distance from 1 to \(n\).
**Corollary**

\[
F_{H_n}(1, n) = \frac{(n - 1)F_{2n-2} + 4F^2_{n-1}}{5}.
\]

Moreover, this is the maximum number of spanning 2-forests separating two vertices in the graph.

Dividing by \(F_{2n-2}\) gives the resistance distance from 1 to \(n\).

Another interesting corollary that follows in a few lines is

\[
\lim_{n \to \infty} [r_{H_{n+1}}(1, n + 1) - r_{H_n}(1, n)] = \frac{1}{5}.
\]
Corollary

\[ F_{H_n}(1, n) = \frac{(n - 1)F_{2n-2} + 4F_{n-1}^2}{5}. \]

Moreover, this is the maximum number of spanning 2-forests separating two vertices in the graph.

Dividing by \( F_{2n-2} \) gives the resistance distance from 1 to \( n \).

Another interesting corollary that follows in a few lines is

\[ \lim_{n \to \infty} \left[ r_{H_{n+1}}(1, n + 1) - r_{H_n}(1, n) \right] = \frac{1}{5}. \]

We don’t know how to prove that the maximum resistance distance occurs between the two degree two vertices, nor establish this limit without the exact formula.
Illustrative Example

Straight Linear 2-tree

Linear 2-tree with a single bend
Illustrative Example

Straight Linear 2-tree

Linear 2-tree with a single bend

The second is obtained from the first by a 2-switch
Straight Linear 2-tree

\[
\begin{bmatrix}
0 & 1597 & 1597 & 2296 & 2741 & 3285 & 3792 & 4317 & 4861 & 5576 \\
1597 & 0 & 1220 & 1453 & 2076 & 2552 & 3085 & 3600 & 4148 & 4861 \\
1597 & 1220 & 0 & 1165 & 1432 & 2044 & 2525 & 3060 & 3600 & 4317 \\
2296 & 1453 & 1165 & 0 & 1157 & 1429 & 2040 & 2525 & 3085 & 3792 \\
2741 & 2076 & 1432 & 1157 & 0 & 1156 & 1429 & 2044 & 2552 & 3285 \\
3285 & 2552 & 2044 & 1429 & 1156 & 0 & 1157 & 1432 & 2076 & 2741 \\
3792 & 3085 & 2525 & 2040 & 1429 & 1157 & 0 & 1165 & 1453 & 2296 \\
4317 & 3600 & 3060 & 2525 & 2044 & 1432 & 1165 & 0 & 1220 & 1597 \\
4861 & 4148 & 3600 & 3085 & 2552 & 2076 & 1453 & 1220 & 0 & 1597 \\
5576 & 4861 & 4317 & 3792 & 3285 & 2741 & 2296 & 1597 & 1597 & 0
\end{bmatrix}
\]
Straight Linear 2-tree

Spanning 2-Forest Matrix

\[ S_{strt} = \begin{bmatrix} 
0 & 1597 & 1597 & 2296 & 2741 & 3285 & 3792 & 4317 & 4861 & 5576 \\
1597 & 0 & 1220 & 1453 & 2076 & 2552 & 3085 & 3600 & 4148 & 4861 \\
1597 & 1220 & 0 & 1165 & 1432 & 2044 & 2525 & 3060 & 3600 & 4317 \\
2296 & 1453 & 1165 & 0 & 1157 & 1429 & 2040 & 2525 & 3085 & 3792 \\
2741 & 2076 & 1432 & 1157 & 0 & 1156 & 1429 & 2044 & 2552 & 3285 \\
3285 & 2552 & 2044 & 1429 & 1156 & 0 & 1157 & 1432 & 2076 & 2741 \\
3792 & 3085 & 2525 & 2040 & 1429 & 1157 & 0 & 1165 & 1453 & 2296 \\
4317 & 3600 & 3060 & 2525 & 2044 & 1432 & 1165 & 0 & 1220 & 1597 \\
4861 & 4148 & 3600 & 3085 & 2552 & 2076 & 1453 & 1220 & 0 & 1597 \\
5576 & 4861 & 4317 & 3792 & 3285 & 2741 & 2296 & 1597 & 1597 & 0 
\]
Linear 2-tree with Single Bend

[Diagram of a linear 2-tree with single bend]

Spanning 2-Forest Matrix

Wayne Barrett (BYU)
Linear 2-tree with Single Bend

Spanning 2-Forest Matrix

\[ S_{bnd} = \begin{bmatrix}
0 & 1597 & 1597 & 2296 & 2741 & 3285 & 3664 & 4029 & 4637 & 5320 \\
1597 & 0 & 1220 & 1453 & 2076 & 2552 & 2973 & 3348 & 3952 & 4637 \\
1597 & 1220 & 0 & 1165 & 1432 & 2044 & 2381 & 2736 & 3348 & 4029 \\
2296 & 1453 & 1165 & 0 & 1157 & 1429 & 1976 & 2381 & 2973 & 3664 \\
2741 & 2076 & 1432 & 1157 & 0 & 1156 & 1157 & 1432 & 2076 & 2741 \\
3285 & 2552 & 2044 & 1429 & 1156 & 0 & 1429 & 2044 & 2552 & 3285 \\
3664 & 2973 & 2381 & 1976 & 1157 & 1429 & 0 & 1165 & 1453 & 2296 \\
4029 & 3348 & 2736 & 2381 & 1432 & 2044 & 1165 & 0 & 1220 & 1597 \\
4637 & 3952 & 3348 & 2973 & 2076 & 2552 & 1453 & 1220 & 0 & 1597 \\
5320 & 4637 & 4029 & 3664 & 2741 & 3285 & 2296 & 1597 & 1597 & 0 \\
\end{bmatrix} \]
\[
S_{strt} = \begin{bmatrix}
0 & 1597 & 1597 & 2296 & 2741 & 3285 & 3792 & 4317 & 4861 & 5576 \\
1597 & 0 & 1220 & 1453 & 2076 & 2552 & 3085 & 3600 & 4148 & 4861 \\
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2296 & 1453 & 1165 & 0 & 1157 & 1429 & 2040 & 2525 & 3085 & 3792 \\
2741 & 2076 & 1432 & 1157 & 0 & 1156 & 1429 & 2044 & 2552 & 3285 \\
3285 & 2552 & 2044 & 1429 & 1156 & 0 & 1157 & 1432 & 2076 & 2741 \\
3792 & 3085 & 2525 & 2040 & 1429 & 1157 & 0 & 1165 & 1453 & 2296 \\
4317 & 3600 & 3060 & 2525 & 2044 & 1432 & 1165 & 0 & 1220 & 1597 \\
4861 & 4148 & 3600 & 3085 & 2552 & 2076 & 1453 & 1220 & 0 & 1597 \\
5576 & 4861 & 4317 & 3792 & 3285 & 2741 & 2296 & 1597 & 1597 & 0
\end{bmatrix}
\]

\[
S_{bnd} = \begin{bmatrix}
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1597 & 0 & 1220 & 1453 & 2076 & 2552 & 2973 & 3348 & 3952 & 4637 \\
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2741 & 2076 & 1432 & 1157 & 0 & 1156 & 1157 & 1432 & 2076 & 2741 \\
3285 & 2552 & 2044 & 1429 & 1156 & 0 & 1429 & 2044 & 2552 & 3285 \\
3664 & 2973 & 2381 & 1976 & 1157 & 1429 & 0 & 1165 & 1453 & 2296 \\
4029 & 3348 & 2736 & 2381 & 1432 & 2044 & 1165 & 0 & 1220 & 1597 \\
4637 & 3952 & 3348 & 2973 & 2076 & 2552 & 1453 & 1220 & 0 & 1597 \\
5320 & 4637 & 4029 & 3664 & 2741 & 3285 & 2296 & 1597 & 1597 & 0
\end{bmatrix}
\]
\[
S_{\text{strt}} = \begin{bmatrix}
0 & 1597 & 1597 & 2296 & 2741 & 3285 & 3792 & 4317 & 4861 & 5576 \\
1597 & 0 & 1220 & 1453 & 2076 & 2552 & 3085 & 3600 & 4148 & 4861 \\
1597 & 1220 & 0 & 1165 & 1432 & 2044 & 2525 & 3060 & 3600 & 4317 \\
2296 & 1453 & 1165 & 0 & 1157 & 1429 & 2040 & 2525 & 3085 & 3792 \\
2741 & 2076 & 1432 & 1157 & 0 & 1156 & 1429 & 2044 & 2552 & 3285 \\
3285 & 2552 & 2044 & 1429 & 1156 & 0 & 1157 & 1432 & 2076 & 2741 \\
3792 & 3085 & 2525 & 2040 & 1429 & 1157 & 0 & 1165 & 1453 & 2296 \\
4317 & 3600 & 3060 & 2525 & 2044 & 1432 & 1165 & 0 & 1220 & 1597 \\
4861 & 4148 & 3600 & 3085 & 2552 & 2076 & 1453 & 1220 & 0 & 1597 \\
5576 & 4861 & 4317 & 3792 & 3285 & 2741 & 2296 & 1597 & 1597 & 0 \\
\end{bmatrix}
\]

\[
S_{\text{bnd}} = \begin{bmatrix}
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4637 & 3952 & 3348 & 2973 & 2076 & 2552 & 1453 & 1220 & 0 & 1597 \\
5320 & 4637 & 4029 & 3664 & 2741 & 3285 & 2296 & 1597 & 1597 & 0 \\
\end{bmatrix}
\]
\[ S_{strt} - S_{bnd} = \]
\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 128 & 288 & 224 & 256 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 112 & 252 & 196 & 224 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 144 & 324 & 252 & 288 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 64 & 144 & 112 & 128 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 272 & 612 & 476 & 544 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -272 & -612 & -476 & -544 \\
128 & 112 & 144 & 64 & 272 & -272 & 0 & 0 & 0 & 0 \\
288 & 252 & 324 & 144 & 612 & -612 & 0 & 0 & 0 & 0 \\
224 & 196 & 252 & 112 & 476 & -476 & 0 & 0 & 0 & 0 \\
256 & 224 & 288 & 128 & 544 & -544 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]
\[ S_{strt} - S_{bnd} = \]
\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 128 & 288 & 224 & 256 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 112 & 252 & 196 & 224 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 144 & 324 & 252 & 288 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 64 & 144 & 112 & 128 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 272 & 612 & 476 & 544 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -272 & -612 & -476 & -544 \\
128 & 112 & 144 & 64 & 272 & -272 & 0 & 0 & 0 & 0 \\
288 & 252 & 324 & 144 & 612 & -612 & 0 & 0 & 0 & 0 \\
224 & 196 & 252 & 112 & 476 & -476 & 0 & 0 & 0 & 0 \\
256 & 224 & 288 & 128 & 544 & -544 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[ U = \]
\[
\begin{bmatrix}
128 & 288 & 224 & 256 \\
112 & 252 & 196 & 224 \\
144 & 324 & 252 & 288 \\
64 & 144 & 112 & 128 \\
272 & 612 & 476 & 544 \\
-272 & -612 & -476 & -544 \\
\end{bmatrix}
\]
\[ S_{strt} - S_{bnd} = \]

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 128 & 288 & 224 & 256 \\
0 & 0 & 0 & 0 & 0 & 0 & 112 & 252 & 196 & 224 \\
0 & 0 & 0 & 0 & 0 & 0 & 144 & 324 & 252 & 288 \\
0 & 0 & 0 & 0 & 0 & 0 & 64 & 144 & 112 & 128 \\
0 & 0 & 0 & 0 & 0 & 0 & 272 & 612 & 476 & 544 \\
0 & 0 & 0 & 0 & 0 & 0 & 128 & 288 & 224 & 256 \\
128 & 112 & 144 & 64 & 272 & -272 & 0 & 0 & 0 & 0 \\
288 & 252 & 324 & 144 & 612 & -612 & 0 & 0 & 0 & 0 \\
224 & 196 & 252 & 112 & 476 & -476 & 0 & 0 & 0 & 0 \\
256 & 224 & 288 & 128 & 544 & -544 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[ U = \begin{bmatrix}
128 & 288 & 224 & 256 \\
112 & 252 & 196 & 224 \\
144 & 324 & 252 & 288 \\
64 & 144 & 112 & 128 \\
272 & 612 & 476 & 544 \\
-272 & -612 & -476 & -544
\end{bmatrix} = \begin{bmatrix}
16 \\
14 \\
18 \\
8 \\
34 \\
-34
\end{bmatrix} \begin{bmatrix}
8 & 18 & 14 & 16
\end{bmatrix}\]
\[
\begin{bmatrix}
16 \\
14 \\
18 \\
8 \\
34 \\
-34
\end{bmatrix}
\]
\[
\begin{bmatrix}
16 \\
14 \\
18 \\
8 \\
34 \\
-34
\end{bmatrix} - 16 \begin{bmatrix} 1 \\
1 \\
1 \\
1 \\
1 \\
1 \end{bmatrix}
\]
\[ \begin{bmatrix} 16 \\ 14 \\ 18 \\ 8 \\ 34 \\ -34 \end{bmatrix} - 16 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 2 \\ -8 \\ 18 \\ -50 \end{bmatrix} \]
\[
\begin{bmatrix}
16 \\
14 \\
18 \\
8 \\
34 \\
-34
\end{bmatrix} - 16 \begin{bmatrix}
1 \\
1 \\
1 \\
1 \\
1 \\
1
\end{bmatrix} = \begin{bmatrix}
0 \\
-2 \\
2 \\
-8 \\
18 \\
-50
\end{bmatrix} = 2 \begin{bmatrix}
0 \\
-1 \\
1 \\
-4 \\
9 \\
-25
\end{bmatrix}
\]
\[
\begin{bmatrix}
16 \\
14 \\
18 \\
8 \\
34 \\
-34
\end{bmatrix} - 16
\begin{bmatrix}
1 \\
1 \\
1 \\
1 \\
1 \\
1
\end{bmatrix} =
\begin{bmatrix}
0 \\
-2 \\
2 \\
-8 \\
18 \\
-50
\end{bmatrix} = 2
\begin{bmatrix}
0 \\
-1 \\
1 \\
-4 \\
9 \\
-25
\end{bmatrix} = 2
\begin{bmatrix}
0^2 \\
-1^2 \\
1^2 \\
-2^2 \\
3^2 \\
-5^2
\end{bmatrix}
\]
\[
\begin{bmatrix}
16 & 14 \\
18 & 8 \\
34 & -34
\end{bmatrix}
- 16
\begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}
=
\begin{bmatrix}
0 \\
-2 \\
-8
\end{bmatrix}
= 2
\begin{bmatrix}
0 \\
-1 \\
-4
\end{bmatrix}
= 2
\begin{bmatrix}
0^2 \\
-1^2 \\
-2^2
\end{bmatrix}
= 2
\begin{bmatrix}
F_0^2 \\
-F_1^2 \\
-F_2^2
\end{bmatrix}.
\]
\[
\begin{bmatrix}
16 \\
14 \\
18 \\
8 \\
34 \\
-34
\end{bmatrix} - 16 \begin{bmatrix}
1 \\
1 \\
1 \\
1 \\
1 \\
1
\end{bmatrix} = \begin{bmatrix}
0 \\
-2 \\
2 \\
-8 \\
18 \\
-50
\end{bmatrix} = 2 \begin{bmatrix}
0 \\
-1 \\
1 \\
-4 \\
9 \\
-25
\end{bmatrix} = 2 \begin{bmatrix}
0^2 \\
-1^2 \\
1^2 \\
-2^2 \\
3^2 \\
-5^2
\end{bmatrix} = 2 \begin{bmatrix}
F_0^2 \\
-F_1^2 \\
F_2^2 \\
-F_3^2 \\
F_4^2 \\
-F_5^2
\end{bmatrix}.
\]

\[
\begin{bmatrix}
8 & 18 & 14 & 16
\end{bmatrix} - 16 \begin{bmatrix}
1 & 1 & 1 & 1 & 1
\end{bmatrix}
\]
\[
\begin{bmatrix}
16 \\ 14 \\ 18 \\ 8 \\ -34
\end{bmatrix}
- 16
\begin{bmatrix}
1 \\ 1 \\ 1 \\ 1 \\ 1
\end{bmatrix}
= \begin{bmatrix}
0 \\ -2 \\ 2 \\ -8 \\ 18
\end{bmatrix}
= 2
\begin{bmatrix}
0 \\ -1 \\ 1 \\ -4 \\ 9
\end{bmatrix}
= 2
\begin{bmatrix}
0^2 \\ -1^2 \\ 1^2 \\ -2^2 \\ 3^2 \\ -5^2
\end{bmatrix}
= 2
\begin{bmatrix}
F^2_0 \\ -F^2_1 \\ F^2_2 \\ -F^2_3 \\ F^2_4 \\ -F^2_5
\end{bmatrix}.
\]

\[
\begin{bmatrix}
8 & 18 & 14 & 16
\end{bmatrix}
- 16
\begin{bmatrix}
1 & 1 & 1 & 1
\end{bmatrix}
= \begin{bmatrix}
-8 & 2 & -2 & 0
\end{bmatrix}
\]
\[
\begin{bmatrix}
16 \\
14 \\
18 \\
8 \\
34 \\
-34
\end{bmatrix}
- 16 \begin{bmatrix}
1 \\
1 \\
1 \\
1 \\
1 \\
1
\end{bmatrix}
= \begin{bmatrix}
0 \\
-2 \\
2 \\
-8 \\
18 \\
-50
\end{bmatrix}
= 2 \begin{bmatrix}
0 \\
-1 \\
1 \\
-4 \\
9 \\
-25
\end{bmatrix}
= 2 \begin{bmatrix}
0^2 \\
-1^2 \\
1^2 \\
-2^2 \\
3^2 \\
-5^2
\end{bmatrix}
= 2 \begin{bmatrix}
F_0^2 \\
-F_1^2 \\
F_2^2 \\
-F_3^2 \\
F_4^2 \\
-F_5^2
\end{bmatrix}.
\]

\[
\begin{bmatrix}
8 & 18 & 14 & 16 \\
\end{bmatrix}
- 16 \begin{bmatrix}
1 & 1 & 1 & 1
\end{bmatrix}
= \begin{bmatrix}
-8 & 2 & -2 & 0
\end{bmatrix}
= 2 \begin{bmatrix}
-4 & 1 & -1 & 0
\end{bmatrix}
\]
\[
\begin{bmatrix}
16 \\
14 \\
18 \\
8 \\
34 \\
-34
\end{bmatrix}
- 16
\begin{bmatrix}
1 \\
1 \\
1 \\
1 \\
1 \\
1
\end{bmatrix}
= \begin{bmatrix}
0 \\
-2 \\
2 \\
-8 \\
18 \\
-50
\end{bmatrix}
= 2
\begin{bmatrix}
0 \\
-1 \\
1 \\
-4 \\
9 \\
-25
\end{bmatrix}
= 2
\begin{bmatrix}
0^2 \\
-1^2 \\
1^2 \\
-2^2 \\
3^2 \\
-5^2
\end{bmatrix}
= 2
\begin{bmatrix}
F_0^2 \\
-F_1^2 \\
F_2^2 \\
-F_3^2 \\
F_4^2 \\
-F_5^2
\end{bmatrix}.
\]

\[
\begin{bmatrix}
8 & 18 & 14 & 16
\end{bmatrix}
- 16
\begin{bmatrix}
1 & 1 & 1 & 1
\end{bmatrix}
= \begin{bmatrix}
-8 \\
2 \\
-2 \\
0
\end{bmatrix}
= 2
\begin{bmatrix}
-4 & 1 & -1 & 0
\end{bmatrix}
= 2
\begin{bmatrix}
-F_3^2 & F_2^2 & -F_1^2 & F_0^2
\end{bmatrix}.
\]
\[
\begin{bmatrix}
16 \\
14 \\
18 \\
8 \\
34 \\
-34
\end{bmatrix}
- 16
\begin{bmatrix}
1 \\
1 \\
1 \\
1 \\
1 \\
1
\end{bmatrix}
= 2
\begin{bmatrix}
0 \\
-2 \\
2 \\
-8 \\
18 \\
-50
\end{bmatrix}
= 2
\begin{bmatrix}
0 \\
-1 \\
1 \\
-4 \\
9 \\
-25
\end{bmatrix}
= 2
\begin{bmatrix}
0^2 \\
-1^2 \\
1^2 \\
-2^2 \\
3^2 \\
-5^2
\end{bmatrix}
= 2
\begin{bmatrix}
F_0^2 \\
-F_1^2 \\
F_2^2 \\
-F_3^2 \\
F_4^2 \\
-F_5^2
\end{bmatrix}.
\]

\[
\begin{bmatrix}
8 & 18 & 14 & 16
\end{bmatrix}
- 16
\begin{bmatrix}
1 & 1 & 1 & 1
\end{bmatrix}
= \begin{bmatrix}
-8 & 2 & -2 & 0
\end{bmatrix}
= 2
\begin{bmatrix}
-4 & 1 & -1 & 0
\end{bmatrix}
= 2
\begin{bmatrix}
-F_3^2 & F_2^2 & -F_1^2 & F_0^2
\end{bmatrix}.
\]

16 = 2 \cdot 8 = F_3 F_6
\[
\begin{bmatrix}
16 \\
14 \\
18 \\
8 \\
34 \\
-34
\end{bmatrix}
- 16
\begin{bmatrix}
1 \\
1 \\
1 \\
1 \\
1 \\
1
\end{bmatrix}
= \begin{bmatrix}
0 \\
-2 \\
2 \\
-8 \\
18 \\
-50
\end{bmatrix}
= 2
\begin{bmatrix}
0 \\
-1 \\
1 \\
-4 \\
9 \\
-25
\end{bmatrix}
= 2
\begin{bmatrix}
0^2 \\
-1^2 \\
1^2 \\
-2^2 \\
3^2 \\
-5^2
\end{bmatrix}
= 2
\begin{bmatrix}
F_0^2 \\
-F_1^2 \\
F_2^2 \\
-F_3^2 \\
F_4^2 \\
-F_5^2
\end{bmatrix}.
\]

\[
[8 \ 18 \ 14 \ 16] - 16 [1 \ 1 \ 1 \ 1] = [-8 \ 2 \ -2 \ 0]
= 2 [-4 \ 1 \ -1 \ 0] = 2 [-F_3^2 \ F_2^2 \ -F_1^2 \ F_0^2].
\]

\[
16 = 2 \cdot 8 = F_3 F_6 \text{ so the } u, v \text{ entry of }
\begin{bmatrix}
16 \\
14 \\
18 \\
8 \\
34 \\
-34
\end{bmatrix}
[8 \ 18 \ 14 \ 16]
\text{ is }

[F_3 F_6 + 2(-1)^{u-1} F_{u-1}^2][F_3 F_6 + 2(-1)^{n-v} F_{n-v}^2].
\]
We incorporate the location of the bend $k$ to generalize to arbitrary size.
We incorporate the location of the bend $k$ to generalize to arbitrary size.

**Theorem**

Let $H_n$ be

![Diagram of $H_n$]

and let $G_n$ be

![Diagram of $G_n$]
Straight versus bent linear 2-tree

We incorporate the location of the bend $k$ to generalize to arbitrary size.

**Theorem**

Let $H_n$ be

![Diagram of H_n](image1)

and let $G_n$ be

![Diagram of G_n](image2)

Then for any two vertices $u$ and $v$ with $u \leq k + 1$ and $v > k + 1$,

$$F_{G_n}(u, v) = F_{H_n}(u, v) - \left[ F_{k-2}F_{k+1} + 2(-1)^{k-u}F_{u-1}^2 \right] \left[ F_{n-k-2}F_{n-k+1} + 2(-1)^{v-k-1}F_{n-v}^2 \right].$$
Substituting $u = 1$ and $v = n$ in

$$\mathcal{F}_{G_n}(u, v) = \mathcal{F}_{H_n}(u, v) - [F_{k-2}F_{k+1} + 2(-1)^{k-u}F_{u-1}^2] [F_{n-k-2}F_{n-k+1} + 2(-1)^{v-k-1}F_{n-v}^2],$$
Substituting $u = 1$ and $v = n$ in

$$
\mathcal{F}_{G_n}(u, v) = \mathcal{F}_{H_n}(u, v) - \left[ F_{k-2}F_{k+1} + 2(-1)^{k-u}F^2_{u-1} \right] \left[ F_{n-k-2}F_{n-k+1} + 2(-1)^{v-k-1}F^2_{n-v} \right],
$$

we obtain the corollary,

$$
\mathcal{F}_{G_n}(1, n) = \mathcal{F}_{H_n}(1, n) - F_{k-2}F_{k+1}F_{n-k-2}F_{n-k+1}
$$
Substituting \( u = 1 \) and \( v = n \) in

\[
\mathcal{F}_{G_n}(u, v) = \mathcal{F}_{H_n}(u, v) - [F_{k-2}F_{k+1} + 2(-1)^{k-u}F_{u-1}^2] [F_{n-k-2}F_{n-k+1} + 2(-1)^{v-k-1}F_{n-v}^2],
\]

we obtain the corollary,

\[
\mathcal{F}_{G_n}(1, n) = \mathcal{F}_{H_n}(1, n) - F_{k-2}F_{k+1}F_{n-k-2}F_{n-k+1} = \frac{(n - 1)F_{2n-2} + 4F_{n-1}^2}{5} - F_{k-2}F_{k+1}F_{n-k-2}F_{n-k+1}.
\]
2-separation formula for the number of spanning 2-forests

Vertex identification

\( G_{ij} \)

Notice that we lose one edge iff \( i \) and \( j \) are adjacent.

**Theorem**

Let \( G \) be a graph with a 2-separation, with \( i, j \) the two vertices separating the graph, and \( G_1, G_2 \) the two graphs of the separation.

Let \( u \in V(G_1) \) and \( v \in V(G_2) \).

Then

\[
F_{G}(u, v) = F_{G_1/ij}(u, ij) T_{G_2} + F_{G_2/ij}(v, ij) T_{G_1} + F_{G_1}(u, i) F_{G_2}(v, j) - 2 F_{G_1}(u, \{i, j\}) F_{G_2}(v, \{i, j\}).
\]
Vertex identification

Let $G$ be a graph with a 2-separation, with $i$, $j$ the two vertices separating the graph, and $G_1$, $G_2$ the two graphs of the separation. Let $u \in V(G_1)$ and $v \in V(G_2)$.

Then $F_G(u, v) = F_{G_1/ij}(u, ij) T(G_2) + F_{G_2/ij}(v, ij) T(G_1) + F_{G_1}(u, i) F_{G_2}(v, j) + F_{G_1}(u, j) F_{G_2}(v, j) - 2 F_{G_1}(u, \{i, j\}) F_{G_2}(v, \{i, j\})$. 

Wayne Barrett (BYU) Spanning 2-Forests and Resistance Distance, July 11, 2019 29 / 34
2-separation formula for the number of spanning 2-forests

**Vertex identification**

\[ G \quad G/ij \]

Notice that we lose one edge iff \( i \) and \( j \) are adjacent.

**Theorem**

Let \( G \) be a graph with a 2-separation, with \( i, j \) the two vertices separating the graph, and \( G_1, G_2 \) the two graphs of the separation.

Let \( u \in V(G_1) \) and \( v \in V(G_2) \).

Then

\[
F_{G}(u, v) = F_{G_1/ij}(u, ij) T_{G_2} + F_{G_2/ij}(v, ij) T_{G_1} + F_{G_1}(u, i) F_{G_2}(v, j) + F_{G_1}(u, j) F_{G_2}(v, j) - 2 F_{G_1}(u, \{i, j\}) F_{G_2}(v, \{i, j\}).
\]
Vertex identification

Notice that we lose one edge iff $i$ and $j$ are adjacent.
2-separation formula for the number of spanning 2-forests

**Vertex identification**

\[ G_{ij} \]

Notice that we lose one edge iff \( i \) and \( j \) are adjacent.

**Theorem**

*Let \( G \) be a graph with a 2-separation, with \( i, j \) the two vertices separating the graph, and \( G_1, G_2 \) the two graphs of the separation.*
2-separation formula for the number of spanning 2-forests

**Vertex identification**

\[ G \quad G/ij \]

Notice that we lose one edge iff \( i \) and \( j \) are adjacent.

**Theorem**

Let \( G \) be a graph with a 2-separation, with \( i, j \) the two vertices separating the graph, and \( G_1, G_2 \) the two graphs of the separation. Let \( u \in V(G_1) \) and \( v \in V(G_2) \).
2-separation formula for the number of spanning 2-forests

Vertex identification

Notice that we lose one edge iff $i$ and $j$ are adjacent.

Theorem

Let $G$ be a graph with a 2-separation, with $i,j$ the two vertices separating the graph, and $G_1, G_2$ the two graphs of the separation.
Let $u \in V(G_1)$ and $v \in V(G_2)$.
Then

$$\mathcal{F}_G(u, v) =$$
2-separation formula for the number of spanning 2-forests

Vertex identification

Notice that we lose one edge iff $i$ and $j$ are adjacent.

Theorem

Let $G$ be a graph with a 2-separation, with $i, j$ the two vertices separating the graph, and $G_1, G_2$ the two graphs of the separation. Let $u \in V(G_1)$ and $v \in V(G_2)$. Then

$$\mathcal{F}_G(u, v) = \mathcal{F}_{G_1/ij}(u, ij)\mathcal{T}(G_2) + \mathcal{F}_{G_2/ij}(v, ij)\mathcal{T}(G_1)$$
Vertex identification

Notice that we lose one edge iff \(i\) and \(j\) are adjacent.

**Theorem**

Let \(G\) be a graph with a 2-separation, with \(i, j\) the two vertices separating the graph, and \(G_1, G_2\) the two graphs of the separation. Let \(u \in V(G_1)\) and \(v \in V(G_2)\). Then

\[ F_G(u, v) = F_{G_1/ij}(u, ij)T(G_2) + F_{G_2/ij}(v, ij)T(G_1) + F_{G_1}(u, i)F_{G_2}(v, j) + F_{G_1}(u, j)F_{G_2}(v, j) \]
2-separation formula for the number of spanning 2-forests

**Vertex identification**

Notice that we lose one edge iff \( i \) and \( j \) are adjacent.

**Theorem**

Let \( G \) be a graph with a 2-separation, with \( i, j \) the two vertices separating the graph, and \( G_1, G_2 \) the two graphs of the separation.

Let \( u \in V(G_1) \) and \( v \in V(G_2) \).

Then

\[
\mathcal{F}_G(u, v) = \mathcal{F}_{G_1/ij}(u, ij)\mathcal{T}(G_2) + \mathcal{F}_{G_2/ij}(v, ij)\mathcal{T}(G_1) \\
+ \mathcal{F}_{G_1}(u, i)\mathcal{F}_{G_2}(v, j) + \mathcal{F}_{G_1}(u, j)\mathcal{F}_{G_2}(v, j) \\
- 2\mathcal{F}_{G_1}(u, \{i, j\})\mathcal{F}_{G_2}(v, \{i, j\}).
\]
Calculate the number of spanning 2-forests/resistance distances for the triangular grid graph.
Very Open Problems

Calculate the number of spanning 2-forests/resistance distances for the triangular grid graph.

or for a straight linear 3-tree on \( n \) vertices
Very Open Problems

Calculate the number of spanning 2-forests/resistance distances for the triangular grid graph.

or for a straight linear 3-tree on $n$ vertices

The second is the graph whose adjacency matrix has all entries on the first 3 super- and sub-diagonal entries equal to 1 and all other entries equal to 0.
What is the number of spanning trees in a 3-tree on \( n \) vertices?
What is the number of spanning trees in a 3-tree on \( n \) vertices?

<table>
<thead>
<tr>
<th>( n )</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T(3T_n) )</td>
<td>16</td>
<td>75</td>
<td>336</td>
<td>1488</td>
<td>6580</td>
<td>29085</td>
<td>...</td>
</tr>
</tbody>
</table>
### Laplacian Matrix Approach

\[ L_{H_n} = \begin{bmatrix}
2 & -1 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
-1 & 3 & -1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
-1 & -1 & 4 & -1 & -1 & 0 & \cdots & 0 & 0 & 0 \\
0 & -1 & -1 & 4 & -1 & -1 & \cdots & 0 & 0 & 0 \\
0 & 0 & -1 & -1 & 4 & -1 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & -1 & 4 & \cdots & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 4 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & -1 & 3 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & -0 & -1 & 2 \\
\end{bmatrix} \]

Show that \( \det L_{H_n}(1, n) = (n-1)F_{2n-2} + 4F_{2n-1} \)
Laplacian Matrix Approach

\[ L_{H_n} = \begin{bmatrix}
2 & -1 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
-1 & 3 & -1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
-1 & -1 & 4 & -1 & -1 & 0 & \cdots & 0 & 0 & 0 \\
0 & -1 & -1 & 4 & -1 & -1 & \cdots & 0 & 0 & 0 \\
0 & 0 & -1 & -1 & 4 & -1 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & -1 & 4 & \cdots & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 4 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & -1 & 3 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & -0 & -1 & 2 \\
\end{bmatrix} \]

Show that \( \det L_{H_n}(1, n) = \frac{(n-1)F_{2n-2} + 4F_{n-1}^2}{5} \)
References


The Squared Cycle graph on $n$ vertices is the graph with 3 additional edges.

Its Laplacian matrix is the circulant whose first row is

\[
\begin{bmatrix}
4 & -1 & -1 & 0 & \cdots & 0 & -1 & -1
\end{bmatrix}
\]

I was surprised to learn the number of spanning trees is $nF^2_n$.


The Squared Cycle graph on \( n \) vertices is the graph

\[
\begin{array}{c}
1 & 2 & 3 & 4 & 5 & 6 & n - 3 & n - 1 \\
\end{array}
\]

with 3 additional edges.

I was surprised to learn the number of spanning trees is \( nF_2^n \).


Baron, Prodinger, Tichy, Boesch, Wang, Fibonacci Quarterly no. 3, 1985
The Squared Cycle graph on \( n \) vertices is the graph

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 & n - 3 & n - 1 \\
\end{array}
\]

with 3 additional edges.

Its Laplacian matrix is the circulant whose first row is

\[
\begin{bmatrix}
4 & -1 & -1 & 0 & \cdots & 0 & -1 & -1
\end{bmatrix}
\]
The Squared Cycle graph on \( n \) vertices is the graph

with 3 additional edges.

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\[
\begin{bmatrix}
4 & -1 & -1 & 0 & \cdots & 0 & -1 & -1 \\
\end{bmatrix}
\]

I was surprised to learn the number of spanning trees is \( nF_n^2 \).
The Squared Cycle graph on $n$ vertices is the graph

with 3 additional edges.

Its Laplacian matrix is the circulant whose first row is

$$
\begin{bmatrix}
4 & -1 & -1 & 0 & \cdots & 0 & -1 & -1
\end{bmatrix}
$$

I was surprised to learn the number of spanning trees is $nF_n^2$.


Baron, Prodinger, Tichy, Boesch, Wang, Fibonacci Quarterly no. 3, 1985