Math 314 Example Test Problems #1 - Key

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Selected Formulas

Cross Products: $a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$

Quadratic Surfaces in $\mathbb{R}^3$

Cone: $x^2/a^2 + y^2/b^2 = z^2/c^2$
Ellipsoid: $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$
Elliptic Paraboloid: $x^2/a^2 + y^2/b^2 = z/c$
Hyperbolic Paraboloid: $x^2/a^2 - y^2/b^2 = z/c$
Hyperboloid of One Sheet: $x^2/a^2 + y^2/b^2 - z^2/c^2 = 1$
Hyperboloid of Two Sheets: $x^2/a^2 - y^2/b^2 - z^2/c^2 = 1$

Local Behavior of Curves

$T(t) = \frac{r'(t)}{|r'(t)|}$

$\kappa = \frac{|T'(t)|}{|r'(t)|}$

$\kappa = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}$

$\kappa = \frac{|f''(x)|}{[1+(f'(x))^2]^{\frac{3}{2}}}$

$N(t) = \frac{T'(t)}{|T'(t)|}$

Motion in Space

$\mathbf{a} = v'\mathbf{T} + \kappa v^2 \mathbf{N}$

$v' = (\mathbf{r}'(t) \cdot \mathbf{r}''(t))/|\mathbf{r}'(t)|$

$\kappa v^2 = |\mathbf{r}'(t) \times \mathbf{r}''(t)|/|\mathbf{r}'(t)|$

True/False and Multiple Choice

1. Which of the following are consistent with the right hand rule?
   - A. The positive x axis points straight up. The positive y axis points north. The positive z axis points east.
   - B. The positive x axis points straight up. The positive y axis points north. The positive z axis points west.
   - C. The positive x axis points straight up. The positive y axis points east. The positive z axis points north.
   - D. The positive x axis points east. The positive y axis points south. The positive z axis points straight down.
   - E. The positive x axis points south. The positive y axis points straight up. The positive z axis points east.
   - F. The positive x axis points south. The positive y axis points straight down. The positive z axis points east.
2. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|$

   A. True B. False
3. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$

   A. True B. False
4. If θ is the angle between vectors $\mathbf{a}$ and $\mathbf{b}$ then $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\theta)$

   A. True B. False
5. If θ is the angle between vectors $\mathbf{a}$ and $\mathbf{b}$ then $\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\theta)$

   A. True B. False
6. Vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel whenever $\mathbf{a} \times \mathbf{b} = 0$ A. True B. False
7. Which are true?
   - A. $\mathbf{i} \times \mathbf{j} = \mathbf{k}$
   - B. $\mathbf{i} \times \mathbf{j} = -\mathbf{k}$
   - C. $\mathbf{j} \times \mathbf{k} = \mathbf{i}$
   - D. $\mathbf{j} \times \mathbf{k} = -\mathbf{i}$
   - E. $\mathbf{k} \times \mathbf{i} = \mathbf{j}$
   - F. $\mathbf{k} \times \mathbf{i} = -\mathbf{j}$
   - G. $\mathbf{j} \times \mathbf{i} = \mathbf{k}$
   - H. $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$
   - I. $\mathbf{k} \times \mathbf{j} = \mathbf{i}$
   - J. $\mathbf{k} \times \mathbf{j} = -\mathbf{i}$
   - K. $\mathbf{i} \times \mathbf{k} = \mathbf{j}$
   - L. $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$
8. $\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{a}$

   A. True B. False
9. $\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$

   A. True B. False

Answers:
1. BCDF, 2. False. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$ 3. True, 4. True, 5. False. $\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin(\theta)$
6. False. The vectors must be nonzero., 7. ACEHJL, 8. False, 9. False

Free Response

1. Find the length of the parametric space curve $\langle t^2, 4t, 4ln(-t)\rangle$ where $-4 \leq t \leq -1$

   Answer:

   $L = \int_a^b |\mathbf{r}(t)|dt$

   $L = \int_{-4}^{-1} |\langle 2t, 4, 4/t\rangle|dt = \int_{-4}^{-1} \sqrt{(2t)^2 + 4^2 + (4/t)^2}dt = \int_{-4}^{-1} \sqrt{4t^2 + 16 + 16/t^2}dt = \int_{-4}^{-1} |2t - 4/t|dt = \int_{-4}^{-\sqrt{2}} -(2t - 4/t)dt + \int_{-\sqrt{2}}^{-1}(2t - 4/t)dt = (-t^2 + 4ln|t|)\Big|_{-4}^{-\sqrt{2}} + (t^2 - 4ln|t|)\Big|_{-\sqrt{2}}^{-1} = (-2 + 4ln(\sqrt{2})) - (-16 + 4ln(4)) + (1 - 4ln(1)) - (2 - 4ln(\sqrt{2})) = -2 + 2ln(2) + 16 - 8ln(2) + 1 - 2 + 2ln(2) = 13 - 4ln(2)$

2. Simplify the following: $a \cdot [(a + b) \times c] + [c \times (a + b)] \cdot a$

   Answer: $a \cdot [(a+b) \times c] + [c \times (a+b)] \cdot a = a \cdot [(a+b) \times c] - [(a+b) \times c] \cdot a = a \cdot [(a+b) \times c] - a \cdot [(a+b) \times c] = 0$

3. Find three different surfaces which contain the space curve $r(t) = \langle 2t, e^t, e^{2t}\rangle$

   Answers: $y = e^{x/2}$, $z = e^x$, $z = y^2$

4. Find the velocity and position vectors of a particle with the given acceleration, and the given initial velocity and position.

   (a) $\mathbf{a}(t) = \langle 2, 0, 2t\rangle$, $\mathbf{v}(0) = \langle 1, 0, 0\rangle$, $\mathbf{r}(0) = \langle 0, 1, 0\rangle$

   Answer:
   $\mathbf{v}(t) = \int \mathbf{a}(t)dt = \langle \int 2dt, \int 0dt, \int 2tdt\rangle = \langle 2t + C_1, 0 + C_2, t^2 + C_3\rangle$
   Since $\mathbf{v}(0) = \langle 1, 0, 0\rangle$, $C_1 = 1$, $C_2 = 0$, $C_3 = 0$, so $\mathbf{v}(t) = \langle 2t + 1, 0, t^2\rangle$

   $\mathbf{r}(t) = \int \mathbf{v}(t)dt = \langle \int(2t + 1)dt, \int 0dt, \int t^2dt\rangle = \langle t^2 + t + C_1, 0 + C_2, \frac{t^3}{3} + C_3\rangle$
   Since $\mathbf{r}(0) = \langle 0, 1, 0\rangle$, $C_1 = 0$, $C_2 = 1$, $C_3 = 0$, so $\mathbf{r}(t) = \langle t^2 + t, 1, \frac{t^3}{3}\rangle$

   (b) $\mathbf{a}(t) = \sin(t)\mathbf{i} + 2\cos(t)\mathbf{j} + 6t\mathbf{k}$, $\mathbf{v}(0) = -\mathbf{k}$, $\mathbf{r}(0) = \mathbf{j} - 4\mathbf{k}$

   Answer:
   $\mathbf{v}(t) = \int \mathbf{a}(t)dt = \int \sin(t)dt\mathbf{i} + \int 2\cos(t)dt\mathbf{j} + \int 6tdt\mathbf{k} = (-\cos(t) + C_1)\mathbf{i} + (2\sin(t) + C_2)\mathbf{j} + (3t^2 + C_3)\mathbf{k}$
   Since $\mathbf{v}(0) = -\mathbf{k}$, $C_1 = 1$, $C_2 = 0$, $C_3 = 1$, so $\mathbf{v}(t) = (1 - \cos(t))\mathbf{i} + 2\sin(t)\mathbf{j} + (3t^2 + 1)\mathbf{k}$

   $\mathbf{r}(t) = \int \mathbf{v}(t)dt = \int(1 - \cos(t))dt\mathbf{i} + \int 2\sin(t)dt\mathbf{j} + \int(3t^2 + 1)dt\mathbf{k}$
   $= (t - \sin(t) + C_1)\mathbf{i} + (-2\cos(t) + C_2)\mathbf{j} + (t^3 + t + C_3)\mathbf{k}$
   Since $\mathbf{r}(0) = \mathbf{j} - 4\mathbf{k}$, $C_1 = 0$, $C_2 = 3$, $C_3 = -4$, so $\mathbf{r}(t) = (t - \sin(t))\mathbf{i} + (3 - 2\cos(t))\mathbf{j} + (t^3 + t - 4)\mathbf{k}$

5. Where is the curvature maximized for $y = e^x$?

   Answer: $(-\frac{1}{2}\ln(2), 1/\sqrt{2})$

   $\kappa = \frac{|f''(x)|}{[1+(f'(x))^2]^{\frac{3}{2}}} = \frac{e^x}{(1+e^{2x})^{\frac{3}{2}}}$. Note that κ is maximized when $\kappa' = 0$

   $\kappa' = \frac{(1+e^{2x})^{\frac{3}{2}} e^x - e^x \frac{3}{2}\sqrt{1+e^{2x}}2e^{2x}}{(1+e^{2x})^3} = \frac{\sqrt{1+e^{2x}}e^x(1+e^{2x}-3e^{2x})}{(1+e^{2x})^3}$

   Note that this is equal to zero precisely when the numerator is equal to zero, which occurs when $1 = 2e^{2x}$, or when $x = \frac{1}{2}\ln(\frac{1}{2}) = -\frac{1}{2}\ln(2)$. Plugging that in for x in our equation for curvature gives us $\kappa = 1/\sqrt{2}$

6. Find equations for the normal plane and osculating plane of the parametric curve $\mathbf{r}(t) = \langle ln(t), 2t, t^2\rangle$ at the point $(0, 2, 1)$.

   Answer:

   Recall that the normal plane contains the normal and binormal vectors, so the normal vector to the normal plane is the tangent vector. Similarly, the osculating plane contains the tangent and normal vectors, so the normal vector to the osculating plane is the binormal vector.

   Normal plane: Finding the tangent vector: $\mathbf{T}(t) = \frac{r'(t)}{|r'(t)|} = \frac{\langle 1/t,2,2t\rangle}{|\langle 1/t,2,2t\rangle|} = \frac{\langle 1/t,2,2t\rangle}{\sqrt{1/t^2+4+4t^2}} = \frac{1}{2t+1/t}\langle 1/t, 2, 2t\rangle$

   The point $(0, 2, 1)$ occurs at $t = 1$, so our tangent vector is $\frac{1}{3}\langle 1, 2, 2\rangle = \langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3}\rangle$ so the normal plane is $\frac{1(x-0)}{3} + \frac{2(y-2)}{3} + \frac{2(z-1)}{3} = 0$

   Osculating plane: Finding the binormal vector: $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$

   $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{\langle -1/t^2,0,2\rangle}{|\langle -1/t^2,0,2\rangle|} = \frac{\langle -1/t^2,0,2\rangle}{\sqrt{4+1/t^4}}$ so $\mathbf{N}(1) = \langle -\frac{1}{\sqrt{5}}, 0, \frac{2}{\sqrt{5}}\rangle$, so $\mathbf{B}(1) = \langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3}\rangle \times \langle -\frac{1}{\sqrt{5}}, 0, \frac{2}{\sqrt{5}}\rangle = \langle \frac{4}{3\sqrt{5}} \frac{-4}{3\sqrt{5}} \frac{2}{3\sqrt{5}}\rangle$ so the osculating plane is $\frac{4(x-0)}{3\sqrt{5}} + \frac{-4(y-2)}{3\sqrt{5}} + \frac{2(z-1)}{3\sqrt{5}} = 0$

7. Find the curvature of the following:

   (a) $\mathbf{r}(t) = t^3\mathbf{j} + t^2\mathbf{k}$

   Answer: $\kappa = \frac{6t^2}{(9t^4+4t^2)^{3/2}}$

   (b) $\mathbf{r}(t) = \sqrt{6}t^2\mathbf{i} + 2t\mathbf{j} + 2t^3\mathbf{k}$

   Answer: $\kappa = \frac{\sqrt{6}}{2(3t^2+1)^2}$

   (c) $\mathbf{r}(t) = \langle t, t^2, t^3\rangle$ at the point (1,1,1).

   Answer: $\kappa = \frac{1}{7}\sqrt{\frac{19}{14}}$

   (d) $y = x^4$

   Answer: $\kappa = \frac{12x^2}{(1+16x^6)^{3/2}}$

   (e) $y = xe^x$

   Answer: $\kappa = \frac{e^x|x+2|}{[1+(xe^x+e^x)^2]^{3/2}}$

   (f) $y = xe^x$ at the point (1,e).

   Answer: $\kappa(1) = \frac{e^1|1+2|}{[1+(1e^1+e^1)^2]^{3/2}} = \frac{3e}{(1+4e^2)^{3/2}}$

8. Find the velocity, acceleration and speed functions of a particle with the given position functions, and evaluate them at the given time t:

(a) $\mathbf{r}(t) = \langle -\frac{1}{2}t^2, t\rangle, \quad t = 2$

Answer:

$\mathbf{v}(t) = \mathbf{r}'(t) = \langle -t, 1\rangle$
$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t) = \langle -1, 0\rangle$
$s(t) = |\mathbf{r}'(t)| = \sqrt{t^2+1}$

$\mathbf{v}(2) = \langle -2, 1\rangle$
$\mathbf{a}(2) = \langle -1, 0\rangle$
$s(2) = \sqrt{2^2+1} = \sqrt{5}$

(b) $\mathbf{r}(t) = 3\cos(t)\mathbf{i} + 2\sin(t)\mathbf{j}, \quad t = \pi/3$

Answer:

$\mathbf{v}(t) = \mathbf{r}'(t) = -3\sin(t)\mathbf{i} + 2\cos(t)\mathbf{j}$
$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t) = -3\cos(t)\mathbf{i} - 2\sin(t)\mathbf{j}$
$s(t) = |\mathbf{r}'(t)| = \sqrt{9\sin^2(t) + 4\cos^2(t)} = \sqrt{4 + 5\sin^2(t)}$

$\mathbf{v}(\pi/3) = -3\sin(\pi/3)\mathbf{i} + 2\cos(\pi/3)\mathbf{j} = -3\frac{\sqrt{3}}{2}\mathbf{i} + \mathbf{j}$
$\mathbf{a}(\pi/3) = -3\cos(\pi/3)\mathbf{i} + -2\sin(\pi/3)\mathbf{j} = -\frac{3}{2}\mathbf{i} - 2\sqrt{3}\mathbf{j}$
$s(\pi/3) = \sqrt{9\sin^2(\pi/3) + 4\cos^2(\pi/3)} = \sqrt{4 + 5\sin^2(\pi/3)} = \sqrt{4 + 15/4} = \frac{\sqrt{31}}{2}$

(c) $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + 2\mathbf{k}, \quad t = 1$

Answer:

$\mathbf{v}(t) = \mathbf{r}'(t) = 1\mathbf{i} + 2t\mathbf{j} + 0\mathbf{k}$
$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t) = 0\mathbf{i} + 2\mathbf{j}$
$s(t) = |\mathbf{r}'(t)| = \sqrt{1+4t^2}$

$\mathbf{v}(1) = 1\mathbf{i} + 2\mathbf{j}$
$\mathbf{a}(1) = 2\mathbf{j}$
$s(1) = \sqrt{5}$

(d) $\mathbf{r}(t) = \sqrt{2}t\mathbf{i} + e^t\mathbf{j} + e^{-t}\mathbf{k}, \quad t = 2$

Answer:

$\mathbf{v}(t) = \mathbf{r}'(t) = \sqrt{2}\mathbf{i} + e^t\mathbf{j} - e^{-t}\mathbf{k}$
$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t) = 0\mathbf{i} + e^t\mathbf{j} + e^{-t}\mathbf{k}$
$s(t) = |\mathbf{r}'(t)| = \sqrt{2 + e^{2t} + e^{-2t}} = e^t + e^{-t}$

$\mathbf{v}(2) = \sqrt{2}\mathbf{i} + e^2\mathbf{j} - e^{-2}\mathbf{k}$
$\mathbf{a}(2) = e^2\mathbf{j} + e^{-2}\mathbf{k}$
$s(2) = e^2 + e^{-2} = e^2 + \frac{1}{e^2}$

(e) $\mathbf{r}(t) = e^t(\cos(t)\mathbf{i} + \sin(t)\mathbf{j} + t\mathbf{k}), \quad t = \pi$

Answer:
$\mathbf{v}(t) = \mathbf{r}'(t) = (-e^t\sin(t) + e^t\cos(t))\mathbf{i} + (e^t\cos(t) + e^t\sin(t))\mathbf{j} + (e^t + te^t)\mathbf{k}$
$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t) = (-2e^t\sin(t))\mathbf{i} + (2e^t\cos(t))\mathbf{j} + (2e^t + te^t)\mathbf{k}$
$s(t) = |\mathbf{r}'(t)| = \sqrt{(-e^t\sin(t) + e^t\cos(t))^2 + (e^t\cos(t) + e^t\sin(t))^2 + (e^t + te^t)^2}$

$\mathbf{v}(\pi) = (-e^\pi\sin(\pi) + e^\pi\cos(\pi))\mathbf{i} + (e^\pi\cos(t) + e^\pi\sin(\pi))\mathbf{j} + (e^\pi + ^\pi)\mathbf{k} = (e^\pi)\mathbf{i} + (e^\pi)\mathbf{j} + (e^\pi + \pi e^\pi)\mathbf{k} = e^\pi(\mathbf{i} + \mathbf{j} + (1+\pi)\mathbf{k})$
$\mathbf{a}(\pi) = (-2e^\pi\sin(\pi))\mathbf{i} + (2e^\pi\cos(\pi))\mathbf{j} + (2e^\pi + \pi e^\pi)\mathbf{k} = 0\mathbf{i} + 2e^\pi\mathbf{j} + (2e^\pi + \pi e^\pi)\mathbf{k}$
$s(\pi) = \sqrt{(-e^\pi\sin(\pi) + e^\pi\cos(\pi))^2 + (e^\pi\cos(\pi) + e^\pi\sin(\pi))^2 + (e^\pi + \pi e^\pi)^2}$

9. Give an equation of the plane perpendicular to the vector $\langle 2, 1, -1\rangle$ which passes through the point $\langle 1, 2, 4\rangle$

Answer: $2(x-1) + 1(y-2) - 1(z-4) = 0$ or $2x + y - z = 0$

10. Give a parametric representation of the line parallel to the x-axis and passes through the point $\langle 3, 4, 5\rangle$

Answer: $\mathbf{r}(t) = \langle 3+t, 4, 5\rangle$

11. Evaluate the following limits, if they exist. If they do not exist, explain why not.

(a)
$$\lim_{(x,y)\to(3,-1)} \frac{x^2y - xy^3 + 5}{x^2 - 3y^2 + xy - 6}$$

Answer: Plugging in $x = 3$ and $y = -1$ gives us 1/3

(b)
$$\lim_{(x,y)\to(0,0)} \frac{x^4 - 4y^2}{x^2 + 2y^2}$$

Answer: Does not exist. Along the path $x = 0$, the limit goes to -2 but along the path $y = 0$, the limit goes to 0

(c)
$$\lim_{(x,y,z)\to(0,0,0)} \frac{xy + yz}{x^2 + y^2 + z^2}$$

Answer: Does not exist. Along the path $x = 0, y = z$ the limit goes to 1/2 but along the path $y = 0, x = z$ it goes to 0.

(d)
$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2 + y^2}}$$

Answer: Let $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Then the limit becomes $\lim_{r\to 0} \frac{r^2\sin(\theta)\cos(\theta)}{|r|} = 0$

(e)

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}$$

Answer: $\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} = \lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)(\sqrt{x^2+y^2+1}+1)}{x^2+y^2+1-1} = \lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)(\sqrt{x^2+y^2+1}+1)}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2+1}+1) = 2$

(f)

$$\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^4+y^4}$$

Answer: Does not exist. Along the path $x = 0$ it goes to 0, but along the path $y = x^{3/4}$ the limit becomes $\lim_{x\to 0} \frac{x^4}{x^4+x^3} = 1$ through repeated uses of L'Hopital's rule.

(g)

$$\lim_{(x,y,z)\to(\pi,0,1/3)} e^{y^2} \tan(xz)$$

Answer: Plugging in gives us $e^0 \tan(\frac{\pi}{3}) = \sqrt{3}$