Math 314 Test 2, Example Problems #1 - Key

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These questions are not from previous exams. These are questions that cover many, but not necessarily all, of the concepts which could appear on the exams. If you feel there is an error with the solutions, please contact the Math Lab via mathlabupper@mathematics.byu.edu, and we will rectify the mistake.

Selected Formulas

Cross Products: $a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$

Second Derivative Test

Critical points: The point (a, b) is a critical point if $f_x(a, b) = 0$ and $f_y(a, b) = 0$ or they do not exist.

Given $D = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$ and that (a, b) is a critical point, then:
(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
(c) If $D < 0$, then $f(a, b)$ is a saddle point.
(d) If $D = 0$, we know nothing about $f(a, b)$.

True/False and Multiple Choice

1. There is a box with dimensions l, w, h. These dimensions change with time. At time $t = 7s$, the dimensions are $l = 3m$, $w = 2m$, $h = 6m$ and l and w are both increasing at a rate of $2m/s$ while h is decreasing at a rate of $5m/s$. Which of the following are decreasing at $t = 7s$?
   - A. Surface Area
   - B. Volume
   - C. Length of the diagonal
   - D. None of these

2. For the same box in the first question, which of these is the correct magnitude for the rate of change in surface area?
   - A. $9\frac{m^2}{s}$
   - B. $18\frac{m^2}{s}$
   - C. $36\frac{m^2}{s}$
   - D. $59\frac{m^2}{s}$
   - E. $118\frac{m^2}{s}$
   - F. None of these

3. For the same box in the first question, which of these is the correct magnitude for the rate of change in volume?

   A. $15\frac{m^3}{s}$

   B. $30\frac{m^3}{s}$

   C. $45\frac{m^3}{s}$

   D. $60\frac{m^3}{s}$

   E. $75\frac{m^3}{s}$

   F. $90\frac{m^3}{s}$

   G. None of these

4. For the same box in the first question, which of these is the correct magnitude for the rate of change in the length of the diagonal?

   A. $-\frac{20}{7}\frac{m}{s}$

   B. $-\frac{40}{7}\frac{m}{s}$

   C. $-\frac{60}{7}\frac{m}{s}$

   D. $-\frac{80}{7}\frac{m}{s}$

   E. $-\frac{1000}{7}\frac{m}{s}$

   F. None of these

5. Let $z = f(x, y)$ and both x and y are functions of t. Then

$$\frac{dz}{dt} = \frac{dz}{dx}\frac{dx}{dt} = \frac{dz}{dy}\frac{dy}{dt}$$

   A. True B. False

6. If $6xy - \sin(x^2 + y) = x^2y^3$, what is $\frac{dy}{dx}$?

   A. $\frac{dy}{dx} = \frac{2x\sin(x^2+y)+2xy^3-6y}{6x+3x^2y^2+\cos(x^2+y)}$

   B. $\frac{dy}{dx} = \frac{2x\sin(x^2+y)+2xy^3}{6x-3x^2y^2-\cos(x^2+y)(2x+1)}$

   C. $\frac{dy}{dx} = \frac{2x\cos(x^2+y)+2xy^3}{6x+3x^2y^2+\cos(x^2+y)}$

   D. $\frac{dy}{dx} = \frac{2x\cos(x^2+y)+2xy^3-6y}{6x-3x^2y^2-\cos(x^2+y)}$

   E. $\frac{dy}{dx} = \frac{2x\sin(x^2+y)+2xy^3-6y}{6x-3x^2y^2-\cos(x^2+y)(2x+1)}$

   F. None of these

7. What is the directional derivative of $f(x, y, z) = xe^y + ye^z + ze^x$ at the point $(0, 0, 0)$ in the direction $\langle 5, 1, -2\rangle$?

   A. $1/\sqrt{30}(5e^y + 5ze^x + e^z + xe^y - 2e^x - 2ye^z)$

   B. $-1/\sqrt{30}(5e^y + 5ze^x + e^z + xe^y - 2e^x - 2ye^z)$

   C. 0

   D. $4/\sqrt{30}$

   E. $-4/\sqrt{30}$

Answers:
1. C, 2. B. 3. B, 4. A, 5. False. $\frac{dz}{dt} = \frac{dz}{dx}\frac{dx}{dt} + \frac{dz}{dy}\frac{dy}{dt}$, 6. D, 7. D

Free Response

1. Let $f(x) = \cos(xy) - \sin(x)e^y$. Calculate f_x, f_y, f_{xx}, f_{yy}, f_{xy}, and f_{yx}.

   Answer: $f_x = -\sin(xy)y - \cos(x)e^y$

   $f_y = -\sin(xy)x - \sin(x)e^y$

   $f_{xx} = -\cos(xy)y^2 + \sin(x)e^y$

   $f_{yy} = -\cos(xy)x^2 - \sin(x)e^y$

   $f_{xy} = f_{yx} = -\sin(xy) - \cos(xy)xy - \cos(x)e^y$

2. Find the volume of the solid enclosed by the parabolic cylinders $y = 1 - x^2$, $y = x^2 - 1$ and the planes $x + y + z = 2$, $2x + 2y - z + 10 = 0$.

   Answer:

$$\int_{-1}^{1}\int_{x^2-1}^{1-x^2} (2x+2y+10) - (2-x-y)dydx = \int_{-1}^{1}\int_{x^2-1}^{1-x^2} 3x+3y+8dydx =$$

$$\int_{-1}^{1} (3x+8)((1-x^2)-(x^2+1)) + \frac{3}{2}((1-x^2)^2 - (x^2-1)^2)dx =$$

$$\int_{-1}^{1} (3x+8)(2-2x^2) + \frac{3}{2}(1-2x^2+x^4-x^4+2x^2-1))dx = \int_{-1}^{1} 6x+16-6x^3-16x^2dx =$$

$$3x^2 + 16x - \frac{3}{2}x^4 - \frac{16}{3}x^3|_{-1}^{1} = (3+16-\frac{3}{2}-\frac{16}{3}) - (3-16-\frac{3}{2}+\frac{16}{3}) = 32 - \frac{32}{3} = \frac{64}{3}$$

3. Using polar coordinates, evaluate

$$\int\int_R ye^x dA$$

   where R is the region in the first quadrant enclosed by the circle $x^2 + y^2 = 25$.

   Answer:

$$\int\int_R ye^x dA = \int_0^{\pi/2}\int_0^5 r\sin(\theta)e^{r\cos(\theta)}rdrd\theta =$$

   Let $u = r\cos(\theta)$. $du = -r\sin(\theta)d\theta$. Thus it becomes

$$\int_0^5\int_r^0 -re^u dudr = \int_0^5 r(e^r - 1)dr = (5e^5 - e^5 - \frac{5^2}{2}) - (0 - 1 - 0) = 4e^5 - \frac{23}{2}$$

4. Find equations for the tangent planes for the following curves at the specified points.

   (a) $z = x^2 \ln|x+y|$ at $(3, -2, 0)$
   Answer: Tangent planes are of the form $z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$
   Thus, since $f_x = x^2/(x+y) + 2x \ln|x+y|$ and $f_y = x^2/(x+y)$, our tangent plane is
   $z = f_x(3, -2)(x-3) + f_y(3, -2)(y+2) = (9 + 6\ln 1)(x-3) + 9(y+2) = 9x + 9y - 9$

   (b) $z = e^{x^2+y^2-1}$ at $(1, 0, 1)$
   Answer: Tangent planes are of the form $z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$
   Thus, since $f_x = 2xe^{x^2+y^2-1}$ and $f_y = 2ye^{x^2+y^2-1}$, our tangent plane is
   $z - 1 = f_x(1, 0)(x-1) + f_y(1, 0)(y) = 2(x-1) + 0(y-0)$, or $z = 2x - 1$

   (c) $z = \cos((x^2+y^2)\frac{\pi}{5})$ at $(4, -3, 1)$
   Answer: Tangent planes are of the form $z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$
   Thus, since $f_x = -\frac{2\pi x}{5}\sin((x^2+y^2)\frac{\pi}{5})$ and $f_y = -\frac{2\pi y}{5}\sin((x^2+y^2)\frac{\pi}{5})$, our tangent plane is
   $z - 1 = f_x(4, -3)(x-4) + f_y(4, -3)(y+3) = 0(x-4) + 0(y+3) = 0$ or $z = 1$

5. Using the tangent planes found in the previous problem, estimate the z-coordinate of the following points of their respective curves.

   (a) $z = x^2 \ln|x+y|$ at $(2.95, -2.1, z)$
   Answer: The tangent plane is $z = 9x + 9y - 9$, so $z \approx 9(2.95) + 9(-2.1) - 9 = 26.85 - 18.9 - 9 = -1.05$

   (b) $z = e^{x^2+y^2-1}$ at $(1.1, -0.2, z)$
   Answer: The tangent plane is $z = 2x - 1$, so $z \approx 2.2 - 1 = 1.2$

   (c) $z = \cos((x^2+y^2)\frac{\pi}{5})$ at $(4.2, -3.1, z)$
   Answer: The tangent plane is $z = 1$, so $z \approx 1$

6. Evaluate $\int\int_R (5-x)dA$, where $R = \{(x,y) : 0 \le x \le 5, 0 \le y \le 3\}$.

Answer:

$$\int\int_R (5-x)dA = \int_0^3 \int_0^5 (5-x)dxdy = \int_0^3 (5x - \frac{x^2}{2})|_0^5 dy = \int_0^3 \frac{25}{2} dy = \frac{75}{2}$$

7. Find the absolute maximum and minimum values of $f(x,y) = xy^2$ on the set $D = \{(x,y) : x \ge 0, y \ge 0, x^2 + y^2 \le 3\}$.

Answer: Absolute maximums and minimums occur in one of three locations: 1) Critical points in the domain, 2) Critical points along the boundary, or 3) corners of the boundary.

1) Critical Points: $f_x = y^2$, $f_y = 2xy$. $f_x = f_y = 0$ when $y = 0$

2) Boundaries: The curve $x = 0$ gives $f(0,y) = 0$ for all y. The curve $y = 0$ gives $f(x,0) = 0$ for all x. The curve $x^2 + y^2 = 3$ gives $f(x, \sqrt{3-x^2}) = x(3-x^2) = 3x - x^3$. Critical points on this curve are when $3 - 3x^2 = 0$, or when $x = 1$ (because our domain is $0 \le x \le \sqrt{3}$), so $y = \sqrt{2}$.

3) Corners: $(0,0), (0,\sqrt{3}), (\sqrt{3}, 0)$

Evaluating f at each point we found gives a maximum value of 2 at $(1, \sqrt{2})$, and a minimum of 0 found at many points.

8. If $R = [-1,3] \times [0,2]$, use a Riemann sum with $m = 4$, $n = 2$ to estimate the value of $\int\int_R \sin(\pi(x+y)/6)dA$. Take the sample points to be the upper left corners of the squares.

Answer:

$$\int\int_R \sin(\pi(x+y)/6)dA \approx \sum_{x=0}^{3}\sum_{y=0}^{1} \sin(\pi(x+y)/6)(1) =$$

$$\sin(\pi\frac{(0+0)}{6}) + \sin(\pi\frac{(0+1)}{6}) + \sin(\pi\frac{(1+0)}{6}) + \sin(\pi\frac{(1+1)}{6}) +$$

$$\sin(\pi\frac{(2+0)}{6}) + \sin(\pi\frac{(2+1)}{6}) + \sin(\pi\frac{(3+0)}{6}) + \sin(\pi\frac{(3+1)}{6}) =$$

$$\sin(0) + \sin(\frac{\pi}{6}) + \sin(\frac{\pi}{6}) + \sin(\frac{2\pi}{6}) + \sin(\frac{2\pi}{6}) + \sin(\frac{3\pi}{6}) + \sin(\frac{3\pi}{6}) + \sin(\frac{4\pi}{6}) = 3 + \frac{3\sqrt{3}}{2}$$

9. Show that

$$0 \leq \int\int_R \sin(\pi x)\cos(\pi y)dA \leq \frac{1}{32}$$

where $R = [0, \frac{1}{4}] \times [\frac{1}{4}, \frac{1}{2}]$ by using the fact that

$$\int\int_D kdA = k(b-a)(d-c)$$

where $D = [a, b] \times [c, d]$ and k is a constant.

Answer: Since $0 \leq \sin(\pi x) \leq \sqrt{2}/2$ on the interval $[0, 1/4]$ and $0 \leq \cos(\pi y) \leq \sqrt{2}/2$ on the interval $[1/4, 1/2]$, we know that

$$\int\int_R (0)dA \leq \int\int_R \sin(\pi x)\cos(\pi y)dA \leq \int\int_R (\sqrt{2}/2)(\sqrt{2}/2)dA$$

Rewriting the first and last integrals in the form $\int_a^b \int_c^d kdA$ allows us to evaluate those integrals. Therefore,

$$0 = 0(\frac{1}{4} - 0)(\frac{1}{2} - \frac{1}{4}) \leq \int\int_R \sin(\pi x)\cos(\pi y)dA \leq (\frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2})(\frac{1}{4} - 0)(\frac{1}{2} - \frac{1}{4}) = \frac{2}{4}\frac{1}{16} = \frac{1}{32}$$

10. Calculate the following integral:

$$\int_0^1 \int_0^1 (u - v)^5 dudv$$

Answer:

$$\int_0^1 \int_0^1 (u-v)^5 dudv = \int_0^1 \frac{(1-v)}{6} - \frac{(0-v)^6}{6} dv = \frac{1}{6}\int_0^1 (1-v)^6 - v^6 dv = \frac{1}{6}\int_0^1 (v-1)^6 - v^6 dv =$$

$$\frac{1}{6}(\frac{(1-1)^7}{7} - \frac{1^7}{7}) - \frac{1}{6}(\frac{(0-1)^7}{7} - \frac{0^7}{7}) = \frac{-1}{42} - \frac{-1}{42} = 0$$

11. Find the mass and center of mass of an object that occupies the region $D = \{(x, y) : 0 \leq y \leq \sin(\pi x/L), 0 \leq x \leq L$ given that the density of the object is given by $\rho(x, y) = y$.

Answer: Mass is found by integrating the density over the region, or in mathematical terms:

$$\int\int_D \rho(x,y)dA = \int_0^L \int_0^{\sin(\pi x/L)} \rho(x,y)dydx = \int_0^L \int_0^{\sin(\pi x/L)} ydydx = \int_0^L \frac{\sin^2(\pi x/L)}{2}dx =$$

$$\int_0^\pi \frac{L}{2\pi}\sin^2(u)du = \frac{L}{2\pi}\int_0^\pi \frac{1-\cos(2u)}{2}du = \frac{L}{4\pi}(\pi) = \frac{L}{4}$$

The center of mass is found by the following integrals.

$$\bar{x} = \frac{1}{m}\int\int_D x\rho(x,y)dA = \frac{L}{2}$$

(Note that this can be found by integrating directly, which will require integration by parts, or by recognizing that there is symmetry involved, so $\bar{x}$ is simply $L/2$.

$$\bar{y} = \frac{1}{m}\int\int_D y\rho(x,y)dA = \int_0^L \int_0^{\sin(\pi x/L)} y^2dydx = \frac{1}{3}\int_0^L \sin^3(\pi x/L)dx = \frac{L}{3\pi}\int_0^\pi \sin^3(u)du =$$

$$\frac{L}{3\pi}\int_0^\pi (1-\cos^2(u))\sin(u)du = \frac{L}{3\pi}\int_{-1}^1 (-1+v^2)dv = \frac{L}{3\pi}(\frac{4}{3}) = \frac{4L}{9\pi}$$

12. Find the average value of $f(x, y) = x^2y$ over the rectangle R whose vertices are (-1,0), (-1,5), (1,5), (1,0).

Answer:

$$\frac{1}{10}\int_0^5 \int_{-1}^1 x^2ydxdy = \frac{1}{10}\int_0^5 \frac{1}{3}(1-(-1))ydy = \frac{1}{30}\int_0^5 2ydy = \frac{1}{30}(5^2-0^2) = \frac{25}{30} = \frac{5}{6}$$

13. Use Lagrange multipliers to find the maximum and minimum values of the following functions:

(a) $f(x, y, z, t) = x + y + z + t$ under the constraint $x^2 + y^2 + z^2 + t^2 = 1$.

Answer:

$\nabla f = \lambda \nabla g = \langle 1, 1, 1, 1 \rangle = \langle 2\lambda x, 2\lambda y, 2\lambda z, 2\lambda t \rangle$ so $x = y = z = t$. Thus $4x^2 = 1$ and $x = \pm 1/2$. Taking $x = 1/2$, we get the maximum value for f which is 2. Taking $x = -1/2$, we get the minimum which is -2.

(b) $f(x, y, z) = \ln(x^2 + 1) + \ln(y^2 + 1) + \ln(z^2 + 1)$ under the constraint $x^2 + y^2 + z^2 = 12$.

Answer:

$\nabla f = \lambda \nabla g = \langle \frac{2x}{x^2+1}, \frac{2y}{y^2+1}, \frac{2z}{z^2+1} \rangle = \langle 2\lambda x, 2\lambda y, 2\lambda z \rangle$, so we get the equations $1/\lambda = x^2 + 1 = y^2 + 1 = z^2 + 1$, so $x^2 = y^2 = z^2$. Thus $3x^2 = 12$, so $x^2 = 4$ and $x, y, z = \pm 2$. In either case, $f(x, y, z) = 3\ln(5)$.

(c) $f(x, y) = xe^y$ under the constraint $x^2 + y^2 = 2$.

Answer:

$\nabla f = \lambda \nabla g = \langle e^y, xe^y \rangle = \langle 2\lambda x, 2\lambda y \rangle$, so we get the equations $2\lambda x = e^y$ and $2\lambda y = xe^y$, so $2\lambda y = x(2\lambda x)$ and thus $y = x^2$. From our constraint, that means that $y^2 + y - 2 = (y+2)(y-1) = 0$, so $y = -2$ or $y = 1$. Since we are dealing with real numbers, $y \neq -2$. Thus our possible options for (x, y) are $(-1, 1)$ and $(1, 1)$. $f(-1, 1) = -e$, a minimum, and $f(1, 1) = e$, a maximum.

14. Find the volume of the solid enclosed y the paraboloid $z = 2 + x^2 + (y-2)^2$ and the planes $z = 1$, $x = 1$, $x = -1$, $y = 0$, and $y = 4$.

Answer:

$$\int_0^4 \int_{-1}^1 [(2 + x^2 + (y-2)^2) - (1)]dxdy = \int_0^4 \int_{-1}^1 1 + x^2 + (y-2)^2 dxdy = \int_0^4 (2 + (\frac{1}{3} - \frac{-1}{3}) + 2(y-2)^2 dy =$$

$$\int_0^4 \frac{8}{3} + 2(y-2)^2 dy = \int_- 2^2 \frac{8}{3} + 2u^2 du = 4(\frac{8}{3}) + \frac{2}{3}(2^3 - (-2)^3) = \frac{32}{3} + \frac{32}{3} = \frac{64}{3}$$