Math 314 Test 3, Example Problems #1 - Key

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These questions are not from previous exams. These are questions that cover many, but not necessarily all, of the concepts which could appear on the exams. If you feel there is an error with the solutions, please contact the Math Lab via mathlabupper@mathematics.byu.edu, and we will rectify the mistake.

Selected Formulas

Cross Products: $a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$

Second Derivative Test

Critical points: The point (a, b) is a critical point if $f_x(a, b) = 0$ and $f_y(a, b) = 0$ or they do not exist.

Given $D = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$ and that (a, b) is a critical point, then:
(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
(c) If $D < 0$, then $f(a, b)$ is a saddle point.
(d) If $D = 0$, we know nothing about $f(a, b)$.

Multiple Choice

1. In the integral $\int_A^B \int_C^D \int_E^F g(x, y, z)dzdxdy$, which one of the following describes the expressions we can replace E with?

   A. Cannot be a function of x, y, or z

   B. Can be a function of x, and only x

   C. Can be a function of y, and only y

   D. Can be a function of z, and only z

   E. Can be a function of x and y, but not z

   F. Can be a function of x and z, but not y

   G. Can be a function of y and z, but not x

   H. Can be a function of x, y, and z

2. Find the Jacobian of the transformation $x = u/v$, $y = v/w$, $z = w/u$.

   A. 0 B. $1/uvw$ C. $-1/uvw$ D. $2/uvw$ E. $-2/uvw$

Answers:
1. E, 2. A

Free Response

1. A particle moves in a velocity field $\mathbf{v}(x, y) = \langle x^2, x + y^2 \rangle$. If it is at position (2,1) at time $t = 3$, estimate its location at time $t = 3.01$.

   Answer: The velocity at $t = 3$ is $\mathbf{v}(2,1) = \langle 2^2, 2 + 1^2 \rangle = \langle 4, 3 \rangle$. Since the change in time is 0.01, our estimate is $(2,1) + 0.01 * \langle 4, 3 \rangle = (2.04, 1.03)$

2. Find the curl and divergence of the vector fields:

   (a)
   $$\mathbf{F}(x, y, z) = \langle \ln(2y + 3z), \ln(x + 3z), \ln(x + 2y) \rangle$$

   Answer:
   $\text{curl}(\mathbf{F}) = \nabla \times \mathbf{F}$, so
   $$\text{curl}(\mathbf{F}) = (\frac{2}{x + 2y} - \frac{3}{x + 3z})\mathbf{i} + (\frac{3}{2y + 3z} - \frac{1}{x + 2y})\mathbf{j} + (\frac{1}{x + 3z} - \frac{2}{2y + 3z})\mathbf{k}$$
   $\text{div}(\mathbf{F}) = \nabla \cdot \mathbf{F}$, so $\text{div}(\mathbf{F}) = 0 + 0 + 0 = 0$

   (b)
   $$\mathbf{F}(x, y, z) = \langle e^x \sin(y), e^y \sin(z), e^z \sin(x) \rangle$$

   Answer:
   $\text{curl}(\mathbf{F}) = \nabla \times \mathbf{F}$, so
   $$\text{curl}(\mathbf{F}) = (0 - e^y \cos(z))\mathbf{i} + (0 - e^z \cos(x))\mathbf{j} + (0 - e^x \cos(y))\mathbf{k}$$
   $\text{div}(\mathbf{F}) = \nabla \cdot \mathbf{F}$, so $\text{div}(\mathbf{F}) = e^x \sin(y) + e^y \sin(z) + e^z \sin(x)$

3. Prove the following identity, assuming that all partial derivatives are continuous, f is a scalar field, and $\mathbf{F}$ is a vector field.
   $$\text{div}(f\mathbf{F}) = f\text{div}\mathbf{F} + \mathbf{F} \cdot \nabla f$$

   Answer: Let $\mathbf{F} = \langle F_1, F_2, F_3 \rangle$
   $$\text{div}(f\mathbf{F}) = \nabla \cdot (f\mathbf{F}) = \frac{\partial}{\partial x}(fF_1) + \frac{\partial}{\partial y}(fF_2) + \frac{\partial}{\partial x}(fF_3) =$$
   $$f\frac{\partial}{\partial x}F_1 + F_1\frac{\partial}{\partial x}f + f\frac{\partial}{\partial y}F_2 + F_2\frac{\partial}{\partial y}f + f\frac{\partial}{\partial z}F_3 + F_3\frac{\partial}{\partial z}f =$$
   $$f(\frac{\partial}{\partial x}F_1 + \frac{\partial}{\partial y}F_2 + \frac{\partial}{\partial z}F_3) + \mathbf{F} \cdot \nabla f = f\text{div}\mathbf{F} + \mathbf{F} \cdot \nabla f$$

4. A thin wire is bent into the shape of a semicircle $x^2 + y^2 = 4, x \geq 0$. If the linear density is a constant k, find the mass and center of mass of the wire.

   Answer: Mass can be found by multiplying k by the length, so if we let $x = 2\cos(t)$ and $y = 2\sin(t)$, we get $m = \int_C \rho(x,y)ds = \int_C kds = k\int_0^\pi \sqrt{(dx/dt)^2 + (dy/dt)^2}dt = k\int_0^\pi \sqrt{4(\sin^2(t) + \cos^2(t))}dt = k\int_0^\pi 2dt = 2k\pi$

   Center of mass: Since it is a semicircle with constant density, the center lies of the y-axis. To find the y-coordinate, we find the average y value, or $\overline{y}$.

$$\overline{y} = \frac{1}{m}\int_C y\rho(x,y)ds = \frac{1}{2k\pi}\int_0^\pi 2\sin(t)k\sqrt{4}dt = \frac{4k}{2k\pi}\int_0^\pi \sin(t)dt = \frac{2}{\pi}(-\cos(\pi) + \cos(0)) = \frac{4}{\pi}$$

   Thus the center of mass is $(0, 4/\pi)$.

5. Evaluate the surface integral $\int\int_S \mathbf{F} \cdot d\mathbf{S}$ for $\mathbf{F}(x,y,z) = x\mathbf{i} + y\mathbf{j} + z^4\mathbf{k}$, where S is the part of the cone $z = \sqrt{x^2 + y^2}$ beneath the plane $z = 1$ with downward orientation.

   Answer: If we let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$, then

$$\int\int_S \mathbf{F} \cdot d\mathbf{S} = \int\int_S \mathbf{F} \cdot \mathbf{n}dS = -1\int\int_D (-P\frac{dz}{dx} - Q\frac{dz}{dy} + R)dA =$$

$$-1\int\int_D (-x\frac{2x}{2\sqrt{x^2+y^2}} - y\frac{2y}{2\sqrt{x^2+y^2}} + z^4)dA = -1\int\int_D (-\frac{x^2+y^2}{\sqrt{x^2+y^2}} + (\sqrt{x^2+y^2})^4)dA =$$

$$-1\int\int_D ((x^2+y^2)^2 - \sqrt{x^2+y^2})dA = -1\int_0^1\int_0^{2\pi} ((r^2)^2 - r)rdrd\theta = -2\pi\int_0^1 (r^5 - r^2)dr =$$

$$-2\pi((\frac{1}{6} - \frac{1}{3}) - (0 - 0)) = \frac{2\pi}{6} = \pi/3$$

6. Determine whether or not $\mathbf{F}$ is a conservative vector field. If it is, find a function f such that $\mathbf{F} = \nabla f$.

$$\mathbf{F}(x,y) = (xy\cos(xy) + \sin(xy))\mathbf{i} + (x^2\cos(xy))\mathbf{j}$$

   Answer: Check if $dQ/dx = dP/dy$.

   $dQ/dx = 2x\cos(xy) - x^2y\sin(xy) = x\cos(xy) - x^2y\sin(xy) + x\cos(xy) = dP/dy$ so it is conservative.

   To find f such that $\mathbf{F} = \nabla f$, we solve $f = \int Pdx = \int Qdy$. $\int Pdx = \int xy\cos(xy) + \sin(xy)dx = x\sin(xy) + g(y)$ and $\int Qdy = x\sin(xy) + h(x)$, so $f(x,y) = x\sin(xy)$ satisfies $\mathbf{F} = \nabla f$

7. If a circle C with radius 1 rolls along the outside of the circle $x^2 + y^2 = 16$, a fixed point P on C traces out a curve called an *epicycloid*, with parametric equations $x = 5\cos(t) - \cos(5t)$, $y = 5\sin(t) - \sin(5t)$. Use Green's Theorem to find the area enclosed by the epicycloid.

   Answer: The area can be found using the following results from Green's Theorem:

$$A = \oint_C xdy = -\oint ydx = \frac{1}{2}\oint xdy - ydx$$

   Thus,

$$A = 1/2\int_0^{2\pi}(5\cos(t) - \cos(5t))(5\cos(t) - 5\cos(5t))dt - (5\sin(t) - \sin(5t))(-5\sin(t) + 5\sin(5t))dt =$$

$$1/2\int_0^{2\pi}[(25\cos^2(t) - 30\cos(t)\cos(5t) + 5\cos^2(5t)) - (-25\sin^2(t) + 30\sin(t)\sin(5t) - 5\sin^2(5t))]dt =$$

$$1/2\int_0^{2\pi}[25 + 5 - 30(\cos(t)\cos(5t) - \sin(t)\sin(5t))]dt = 1/2\int_0^{2\pi}(30 - 30\cos(t + 5t))dt =$$

$$1/2(60t - 5\sin(6t))|_0^{2\pi} = 60\pi$$

8. Find a parametric representation for the plane that passes through the point $(1, 2, -3)$ and contains the vectors $\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$ and $2\mathbf{i} - \mathbf{j} + 5\mathbf{k}$.

   Answer: $\mathbf{r}(u, v) = (1 + 1u + 2v)\mathbf{i} + (2 + 3u - 1v)\mathbf{j} + (-3 - 4u + 5v)\mathbf{k}$

9. Prove the following identity, assuming that all partial derivatives are continuous, and that $\mathbf{F}, \mathbf{G}$ are vector fields.

$$\text{div}(\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot \text{curl}\mathbf{F} - \mathbf{F} \cdot \text{curl}\mathbf{G}$$

   Answer: Let $\mathbf{F} = \langle F_1, F_2, F_3\rangle$ and $\mathbf{G} = \langle G_!, G_2, G_3\rangle$. Then

$$\text{div}(\mathbf{F} \times \mathbf{G}) = \nabla \cdot (\mathbf{F} \times \mathbf{G}) = \nabla \cdot \langle (F_2G_3 - F_3G_2), (F_3G_1 - F_1G_3), (F_1G_2 - F_2G_1)\rangle =$$

$$\frac{\partial}{\partial x}(F_2G_3 - F_3G_2) + \frac{\partial}{\partial y}(F_3G_1 - F_1G_3) + \frac{\partial}{\partial z}(F_1G_2 - F_2G_1) = F_2\frac{\partial G_3}{\partial x} + \frac{\partial F_2}{\partial x}G_3 -$$

$$F_3\frac{\partial G_2}{\partial x} - \frac{\partial F_3}{\partial x}G_2 + F_3\frac{\partial G_1}{\partial x} + \frac{\partial F_3}{\partial x}G_1 - F_1\frac{\partial G_3}{\partial x} - \frac{\partial F_1}{\partial x}G_3 + F_1\frac{\partial G_2}{\partial x} + \frac{\partial F_1}{\partial x}G_2 - F_2\frac{\partial G_1}{\partial x} - \frac{\partial F_2}{\partial x}G_1 =$$

$$\mathbf{G} \cdot \text{curl}(\mathbf{F}) - \mathbf{F} \cdot \text{curl}(\mathbf{G})$$

10. Find the flux through the surface S of the solid bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$ given the following vector field.

$$\mathbf{F}(x, y, z) = (\cos(z) + xy^2)\mathbf{i} + xe^{-z}\mathbf{j} + (\sin(y) + x^2z)\mathbf{k}$$

Answer: The flux through a surface is calculated by the integral $\int \mathbf{F} \cdot d\mathbf{S}$. Using the Divergence Theorem, $\int \mathbf{F} \cdot d\mathbf{S} = \int\int\int_E \text{div}\mathbf{F}dV$. Thus, using the substitutions $x = r\cos(\theta), y = r\sin(\theta)$, the flux is equal to

$$\int\int\int_E \text{div}\mathbf{F}dV = \int\int\int_E (y^2 + 0 + x^2)dV = \int_0^{2\pi}\int_0^2\int_{r^2}^4 (r^2)rdzdrd\theta = \int_0^{2\pi}\int_0^2 (4r^3 - r^5)drd\theta =$$

$$\int_0^{2\pi} (2^4 - \frac{2^6}{6})d\theta = \int_0^{2\pi} (16 - \frac{64}{6})d\theta = 2\pi\frac{32}{6} = \frac{64\pi}{6} = \frac{32\pi}{3}$$

11. Assume that S is a surface that bounds a closed region and that $\mathbf{F}$ has continuous partial derivatives. Prove that $\int\int_S \text{curl}(\mathbf{F}) \cdot d\mathbf{S} = 0$.

Answer: From Stokes' Theorem, we know that $\int\int_S \text{curl}(\mathbf{F}) \cdot d\mathbf{S} = \int_C \mathbf{F}d\mathbf{r}$. Since S is a surface enclosing a closed region, the boundary curve C of S is a point, so $\int_C \mathbf{F}d\mathbf{r} = \int_0^0 \mathbf{F}d\mathbf{r} = 0$

12. Find the volume of the solid that lies within both the cylinder $x^2+y^2 = 1$ and the sphere $x^2+y^2+z^2 = 4$.

Answer:

$$\int_0^1\int_0^{2\pi} 2\sqrt{4 - r^2}rd\theta dr = 2\pi\int_0^1 2r\sqrt{4 - r^2}dr$$

Let $u = 4 - r^2$, $du = -2r$, so

$$2\pi\int_0^1 2r\sqrt{4 - r^2}dr = -2\pi\int_4^3 \sqrt{u}du = 2\pi\int_3^4 \sqrt{u}du = 2\pi(2/3)(4^{3/2} - 3^{3/2}) = \frac{4\pi}{3}(8 - 3\sqrt{3})$$

13. Evaluate $\int\int_S \text{curl}\mathbf{F}\cdot d\mathbf{S}$ where $\mathbf{F}(x,y,z) = 2y\mathbf{i}+e^x\sin(z)\mathbf{j}+x3^y\mathbf{k}$, and S is the hemisphere $x^2+y^2+z^2 = 9, z \geq 0$, oriented upwards.

Answer: From Stokes' Theorem, we know that $\int\int_S \text{curl}(\mathbf{F})\cdot d\mathbf{S} = \int_C \mathbf{F}\cdot d\mathbf{r}$. Since S is oriented upwards, we need a counter-clockwise parametric equation of the circle $x^2+y^2 = 9$. This is $x = 3\cos(t), y = 3\sin(t)$. Thus $\int\int_S \text{curl}(\mathbf{F})\cdot d\mathbf{S} = \int_C \mathbf{F}\cdot d\mathbf{r} =$

$$\int_0^{2\pi} 2(3\sin(t))(-3\sin(t)) + e^{3\cos(t)}\sin(0) + (0)dt = \int_0^{2\pi} -18\sin^2(t)dt = -18\int_0^{2\pi} \frac{1-\cos(2t)}{2} =$$

$$-9\int_0^{2\pi} 1-\cos(2t) = -9(2\pi) = -18\pi$$

14. Show that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sqrt{x^2+y^2+z^2}e^{-(x^2+y^2+z^2)}dxdydz = 2\pi$$

Answer: Making the substitution into spherical coordinates, we get

$$\int_0^{\infty}\int_0^{2\pi}\int_0^{\pi} \rho e^{-\rho^2}\rho^2\sin(\phi)d\phi d\theta d\rho = 2\int_0^{\infty}\int_0^{2\pi} \rho^3 e^{-\rho^2}d\theta d\rho = 4\pi\int_0^{\infty} \rho^3 e^{-\rho^2}d\rho =$$

$$4\pi\int_0^{\infty} \rho^3 e^{-\rho^2}d\rho = 4\pi\frac{1}{2}\int_0^{-\infty} ue^u du = 2\pi\int_0^{-\infty} ue^u du = 2\pi(u-1)e^u|_0^{-\infty} = 2\pi(0-(-1)e^0) = 2\pi$$

15. Sketch the solid whose volume is given by the integral, and then evaluate the integral.

$$\int_0^4\int_0^{2\pi}\int_r^4 rdzd\theta dr$$

Answer: It is a cone of radius 4 and height 4 whose tip is at the origin and base is on the plane $z = 4$. The integral is the volume of this object and is $64\pi/3$

16. Find the area of the surface with parametric equations $x = u^2$, $y = uv$, $z = \frac{1}{2}v^2$, $0 \leq u \leq 1, 0 \leq v \leq 2$

Answer: The parametric representation of the surface is $\mathbf{r}(u,v) = \langle u^2, uv, \frac{1}{2}v^2 \rangle$, so $\mathbf{r}_u = \langle 2u, v, 0 \rangle$ and $\mathbf{r}_v = \langle 0, u, v \rangle$, so the area is given by the integral

$$A(S) = \int\int_D |\mathbf{r}_u \times \mathbf{r}_v| dA = \int_0^2 \int_0^1 \sqrt{(v^2)^2 + (-2uv)^2 + (2u^2)^2} dudv =$$

$$\int_0^2 \int_0^1 \sqrt{(v^2)^2 + 4u^2v^2 + 4(u^2)^2} dudv = \int_0^2 \int_0^1 \sqrt{((v^2) + (2u^2))^2} dudv = \int_0^2 \int_0^1 v^2 + 2u^2 dudv =$$

$$\int_0^2 v^2 + \frac{2}{3} dv = \frac{2^3}{3} + \frac{4}{3} - 0 = 4$$

17. Find the moments of inertia I_x, I_y, I_z, and I_0 for a cube with side length L if one vertex is located at the origin and three edges lie on the coordinate axis. (Density is constant)

Answer: Let ρ represent density. Because we are dealing with a cube whose edges lie on the axis, $I_x = I_y = I_z$, so it suffices to calculate I_x.

$$I_z = I_y = I_x = \int\int\int_E (\text{distance from axis})^2 \rho dV = \rho \int_0^L \int_0^L \int_0^L y^2 + z^2 dxdydz =$$

$$\rho L \int_0^L \int_0^L y^2 + z^2 dydz = \rho L \int_0^L Lz^2 + \frac{L^3}{3} dz = \rho L(L(\frac{L^3}{3} + \frac{L^4}{3})) = \frac{2\rho L^5}{3}$$

$$I_0 = \int\int\int_E (\text{distance from origin})^2 \rho dV = \rho \int_0^L \int_0^L \int_0^L x^2 + y^2 + z^2 dxdydz =$$

$$\rho \int_0^L \int_0^L \frac{L^3}{3} + L(y^2 + z^2) dydz = \rho \int_0^L \frac{2L^4}{3} + L^2 z^2 dz = \frac{3\rho L^5}{3} = \rho L^5$$

18. Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y, z) = x^2 y\mathbf{i} + \frac{1}{3}x^3\mathbf{j} + xy\mathbf{k}$ and C is the curve of intersection of the hyperbolic paraboloid $z = y^2 - x^2$ and the cylinder $x^2 + y^2 = 1$, oriented counterclockwise as viewed from above.

    Answer: $\text{curl}\mathbf{F} = \langle x, -y, x^2 - x^2 \rangle = \langle u\cos(v), -u\sin(v), 0\rangle$, so

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int\int_S \text{curl}\mathbf{F} \cdot d\mathbf{S} = \int\int_D (-x(-2x) - (-y)(2y) + 0)dA = \int\int_D (2x^2 + 2y^2)dA =$$

$$\int_0^1 \int_0^{2\pi} 2r^2 r d\theta dr = 4\pi \int_0^1 r^3 dr = \frac{4\pi}{4} = \pi$$

19. Use Green's Theorem to evaluate the line integral along the given positively oriented curve.

    (a)

$$\int_C y^4 dx + 2xy^3 dy$$

where C is the ellipse $x^2 + 2y^2 = 2$

Answer: Let $x = \sqrt{2}\cos(t), y = \sin(t)$. Thus,

$$\int_C Pdx + Qdy = \int\int_D (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dA = \int\int_D (2y^3 - 4y^3)dA = \int_0^1 \int_0^{2\pi} -2\sin^3(\theta) r d\theta dr =$$

$$-2\int_0^1 \int_0^{2\pi} (1 - \cos^2(\theta))\sin(\theta) r d\theta dr = 2\int_0^1 \int_1^{-1} r(1 - u^2)dudr =$$

$$2\int_0^1 r((-1 - \frac{(-1)^3}{3}) - (1 - \frac{1^3}{3}))dr = \frac{-8}{3}\int_0^1 rdr = \frac{-8}{3}(\frac{1}{2} - 0) = \frac{-4}{3}$$

(b)

$$\int_C (x^2 + y^2)dx + (x^2 - y^2)dy$$

where C is the triangle with vertices $(0, 0)$, $(2, 1)$, and $(0, 1)$.

Answer:

$$\int_C Pdx + Qdy = \int\int_D (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dA = \int\int_D (2x - 2y)dA = \int_0^1 \int_0^{2-2x} (2x - 2y)dydx =$$

$$\int_0^1 2x(2 - 2x) - (2 - 2x)^2 dx = \int_0^1 4x - 4x^2 - (4 - 8x + 4x^2)dx = \int_0^1 12x - 4dx = (6 - 4) - 0 = 2$$

20. Find the mass and center of mass of the solid S bounded by the paraboloid $z = 4x^2 + 4y^2$ and the plane $z = a$, where $a > 0$, if S has constant density K.

Answer:

$$m = \int\int\int_E KdV = K\int_0^{\sqrt{a}/2}\int_0^{2\pi}\int_{4r^2}^{a} 1dzd\theta dr = K\int_0^{\sqrt{a}/2}\int_0^{2\pi} a{-}4r^2 d\theta dr = 2K\pi\int_0^{\sqrt{a}/2} a{-}4r^2 dr =$$

$$2K\pi(\frac{a\sqrt{a}}{2} - \frac{4}{3}(\frac{\sqrt{a}}{2})^3) = 2K\pi(\frac{2a^{3/2}}{6}) = \frac{2K\pi a^{3/2}}{3}$$

The center of mass lies on the z-axis from symmetries. Thus we only need to calculate $\overline{z}$.

$$\overline{z} = \frac{1}{m}\int\int\int_E zKdV = \frac{3}{2K\pi a^{3/2}}K\int_0^{\sqrt{a}/2}\int_0^{2\pi}\int_{4r^2}^{a} zdzd\theta dr = \frac{3}{2\pi a^{3/2}}\int_0^{\sqrt{a}/2}\int_0^{2\pi}\frac{1}{2}(a-4r^2)^2 d\theta dr =$$

$$\frac{6\pi}{4\pi a^{3/2}}\int_0^{\sqrt{a}/2}(a-4r^2)^2 dr = \frac{3}{2a^{3/2}}\int_0^{\sqrt{a}/2} a^2 - 8ar^2 + 16r^4 dr = \frac{3}{2a^{3/2}}(\frac{a^2\sqrt{a}}{2} - 8a\frac{a^{3/2}}{24} + 16\frac{a^{5/2}}{160}) =$$

$$\frac{3}{2}(\frac{a}{2} - \frac{a}{3} + \frac{a}{10}) = \frac{3}{2}\frac{30a - 20a + 6a}{60} = \frac{2}{5}a$$

Thus the center of mass is $(0, 0, \frac{2}{5}a)$.

21. Determine whether or not the vector field $\mathbf{F}(x, y, z) = y\cos(xy)\mathbf{i} + x\cos(xy)\mathbf{j} + \sin(z)\mathbf{k}$ is conservative. If so, find a function f such that $\mathbf{F} = \nabla f$.

Answer: If we write $\mathbf{F}(x, y, z) = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$, we need to check if the following partial derivatives are equal: $\partial P/\partial y = \partial Q/\partial x$, $\partial P/\partial z = \partial R/\partial x$, and $\partial Q/\partial z = \partial R/\partial y$.

$\partial P/\partial y = \partial Q/\partial x = \cos(xy) - yx\sin(xy)$, $\partial P/\partial z = \partial R/\partial x = 0$, and $\partial Q/\partial z = \partial R/\partial y = 0$, so it is conservative.

Integrating $\int Pdx$, $\int Qdy$, and $\int Rdz$ we find that $f = \sin(xy) - \cos(z) + C$

22. Evaluate
$$\int\int\int_E xyzdV$$
where E lies between the spheres $\rho = 2$ and $\rho = 4$ and above the cone $\phi = \pi/3$.

Answer: In spherical coordinates,
$$\int\int\int_E xyzdV = \int_2^4 \int_0^{2\pi} \int_0^{\pi/3} \rho^3 \sin^2(\phi)\cos(\phi)\sin(\theta)\cos(\theta)\rho^2 \sin(\phi) d\phi d\theta d\rho =$$
$$\int_2^4 \int_0^{2\pi} \int_0^{\sqrt{3}/2} \rho^5 u^3 \sin(\theta)\cos(\theta) du d\theta d\rho = \frac{3}{16}\int_2^4 \int_0^{2\pi} \rho^5 \sin(\theta)\cos(\theta) d\theta d\rho =$$
$$\frac{3}{16}\int_2^4 \int_0^{2\pi} \rho^5 \sin(\theta)\cos(\theta) d\theta d\rho = \frac{3}{16}(\sin^2(\theta)|_0^{2\pi})\int_2^4 \rho^5 d\rho = 0$$

23. Evaluate
$$\int\int\int_B (x^2+y^2+z^2)^2 dV$$
where B is the ball with center at the origin and radius 5.

Answer: In spherical coordinates,
$$\int\int\int_B (x^2+y^2+z^2)^2 dV = \int_0^5 \int_0^{2\pi} \int_0^{\pi} (\rho^4)\rho^2 \sin(\phi) d\phi d\theta d\rho = 2\int_0^5 \int_0^{2\pi} \rho^6 d\theta d\rho =$$
$$4\pi \int_0^5 \rho^6 d\rho = 4\pi \frac{5^7}{7} = \frac{312500\pi}{7}$$