An Interesting Family of Symmetric Polynomials
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Abstract. We discuss two natural extremal problems for homogeneous polynomials. These problems have simple solutions for polynomials in one or two variables but become interesting for polynomials in three or more variables. We introduce a family of homogeneous symmetric polynomials in three variables that solve one of these problems and have a number of other interesting properties. For example, their coefficients are integers that can be expressed as sums of binomial coefficients and possess a certain divisibility property. Furthermore, these polynomials are connected in a simple way to a family of polynomials arising as sharp examples in the study of proper polynomial mappings between balls in complex Euclidean space.

1. INTRODUCTION AND STATEMENT OF RESULTS. Many beautiful results in mathematics illustrate the heuristic principle that those objects satisfying some extremal condition possess additional special properties. We encounter an elementary illustration of this principle in calculus when we observe that, among all rectangles of a fixed perimeter, the one of maximum area is a square. The isoperimetric inequality gives a similar but less trivial example. It states that the length $L$ of a closed curve in the plane and the enclosed area $A$ satisfy $4\pi A \leq L^2$. The extremal curve, for which equality is achieved, is the circle. We also find many illustrations of this principle in extremal graph theory. For example, given a fixed number $n$ of vertices, the graph with the maximum number of edges and no clique of size $r + 1$ is the lovely Turán graph $T(n,r)$.

In this article, we explore the properties of a family of symmetric polynomials that arise as solutions to a certain simple extremal problem. Perhaps not surprisingly, in light of the discussion above, their coefficients possess a number of beautiful combinatorial and number-theoretic properties. In order to state the extremal problem, we make several definitions. A polynomial $p$ is said to be symmetric if it is invariant under any permutation of the variables. For example, $p_1(x, y) = x^2 + xy + y^2$ is a symmetric polynomial in two variables because $p_1(y, x) = p_1(x, y)$, but $p_2(x, y) = x^2 + xy - y^2$ is not symmetric.

Let $s$ be the elementary symmetric polynomial of degree 1 in $n$ variables. Thus for $n = 2$, $s(x, y) = x + y$ and for $n = 3$, $s(x, y, z) = x + y + z$. For $k$ a nonnegative integer, let $V_k$ denote the vector space of homogeneous polynomials of degree $k$ in $n$ variables with complex coefficients, together with the zero polynomial. We say $q \in V_k$ is full if every monomial of degree $k$ in $n$ variables appears in $q$ with nonzero coefficient. For example, if $n = 2$, then $q_1(x, y) = x^3 - x^2 y + x y^2 - y^3$ is a full polynomial of degree 3, whereas $q_2(x, y) = x^3 + y^3$ is not full.

For any polynomial $p$, the rank of $p$, denoted by $R(p)$, is the number of distinct monomials appearing in $p$ with nonzero coefficient. We consider the following basic question about homogeneous polynomials:

MSC: Primary 32H35, Secondary 11C08

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**Question 1.** Among all homogeneous symmetric polynomials \( p \) of degree \( m \) with \( p = sq \) for full quotient \( q \), what is the minimum possible rank? What are the polynomials with this minimum rank?

We call such polynomials **extreme symmetric polynomials**. Question 1 is related to an even simpler question (which, itself, has connections to an important problem in several complex variables; see Section 2):

**Question 2.** Among all homogeneous polynomials \( p \) of degree \( m \) with \( p = sq \) for full quotient \( q \), what is the minimum possible rank? What are the polynomials with this minimum rank?

We call such polynomials **sharp polynomials**.

For polynomials in one variable, these questions are uninteresting because any nonzero homogeneous polynomial of degree \( m \) is just a multiple of \( x^m \) and thus has rank 1. For polynomials in two variables, both Questions 1 and 2 have simple answers. For any degree \( m \geq 1 \), there exists a homogeneous polynomial \( p \) with \( p = sq \), with full quotient \( q \), and with only two terms. Indeed, if

\[
p(x, y) = x^m + (-1)^m - 1 y^m,
\]

then

\[
q(x, y) = \frac{p(x, y)}{x + y} = x^{m-1} - x^{m-2} y + \cdots + (-1)^{m-1} y^{m-1},
\]

which is full. Thus \( p \) is a sharp polynomial in two variables. When \( m \) is odd, \( p \) is symmetric, and it is thus also an extreme symmetric polynomial. When \( m \) is even, \( p \) is not symmetric, and, in fact, it is easy to see that there is no homogeneous symmetric polynomial divisible by \( x + y \) having only two terms. We can, however, find such a polynomial with three terms; for \( m = 2r \),

\[
p(x, y) = x^{2r} + 2(-1)^{r-1} x^r y^r + y^{2r}
\]

is such a polynomial. One verifies that its quotient when divided by \( x + y \) is

\[
(x^r + (-1)^{-1} y^r) \sum_{j=0}^{r-1} (-1)^j x^{r-1-j} y^j,
\]

which is full. Thus for \( m \) even, the extreme symmetric polynomials in two variables have three terms.

In this article, we explore the above questions for polynomials in three variables. As far as we know, the questions have not been considered for polynomials in more than three variables. In Section 2, we describe previous research that has identified a family \( \{ F_m \} \) of sharp polynomials with surprising properties and with connections to questions in several complex variables. We also describe a new family \( \{ S_m \} \) of extreme symmetric polynomials (one for each positive degree \( m \)) in three variables. The first few odd-degree members of the family are

\[
S_1 = x + y + z \\
S_3 = x^3 + y^3 + z^3 - 3xyz \\
S_5 = x^3 + y^5 + z^5 + 5xyz(xy + xz + yz)
\]
\(S_7 = x^7 + y^7 + z^7 - 7xyz(x^2y^2 + x^2z^2 + y^2z^2)\)
\(S_9 = x^9 + y^9 + z^9 + 9xyz(x^3y^3 + x^3z^3 + y^3z^3) - 30x^3y^3z^3\)
\(S_{11} = x^{11} + y^{11} + z^{11} - 11xyz(x^4y^4 + x^4z^4 + y^4z^4) + 55x^3y^3z^3(xy + xz + yz)\)
\(S_{13} = x^{13} + y^{13} + z^{13} + 13xyz(x^5y^5 + x^5z^5 + y^5z^5) - 91x^3y^3z^3(x^2y^2 + x^2z^2 + y^2z^2),\)

and the first few even-degree members are
\(S_2 = x^2 + y^2 + z^2 + 2(xy + xz + yz)\)
\(S_4 = x^4 + y^4 + z^4 - 2(x^2y^2 + x^2z^2 + y^2z^2)\)
\(S_6 = x^6 + y^6 + z^6 + 2(x^3y^3 + x^3z^3 + y^3z^3) - 9x^2y^2z^2\)
\(S_8 = x^8 + y^8 + z^8 - 2(x^4y^4 + x^4z^4 + y^4z^4) + 16x^2y^2z^2(xy + xz + yz).\)

We give general formulas for the \(S_m\) below. In addition to this symbolic representation of these polynomials, we can give a visual representation by using a Newton diagram. Although it is possible to construct a Newton diagram for a homogeneous polynomial in any number of variables, we will only consider diagrams for polynomials in three variables. The Newton diagram for a homogeneous polynomial \(g(x, y, z)\) is a graph with one vertex for each monomial appearing in \(g\) with nonzero coefficient. We introduce an edge between the vertices associated with two monomials \(m_1\) and \(m_2\) if there are two unequal degree one monomials \(\lambda_1\) and \(\lambda_2\) such that \(\lambda_1m_1 = \lambda_2m_2\). We illustrate by constructing the Newton diagram for the degree three homogeneous polynomial \(g(x, y, z) = x^3 + 3x^2y - 2x^2z + xyz - z^3\). The Newton diagram will have five vertices because there are five nonzero coefficients; see Figure 1.

![Figure 1. Newton diagram for \(g(x, y, z)\).](image-url)

We find the Newton diagrams easier to read if we simply label each vertex with the coefficient of the corresponding term and if we superimpose the diagram on a dashed grid that shows all possible monomials of the degree of our homogeneous polynomial. We always orient the diagram so that higher rows correspond to higher powers of \(z\), higher powers of \(x\) are to the left, and higher powers of \(y\) are to the right. Figure 2 shows the Newton diagram for \(g\) again, incorporating these conventions.
The Newton diagrams for the extreme symmetric polynomials are particularly lovely. These diagrams are quite sparse, of course, because the extreme symmetric polynomials are, by definition, the symmetric polynomials of minimum rank divisible by \(x + y + z\) with full quotient. Figures 3, 4, and 5 show the Newton diagrams for \(S_3\), \(S_5\), and \(S_7\).

Our main theorem, of course, states that the polynomials \(S_m\) satisfy one or both of the extremal conditions.

**Theorem 1.** If \(S_m\) is defined as in (2)–(5), then \(S_m\) is an extreme symmetric polynomial. Furthermore, if \(m \equiv 1, 3 \mod 6\), then \(S_m\) is a sharp polynomial as well.

**Remark 1.** If \(m\) is not congruent to 1 or 3 \(\mod 6\), then \(S_m\) is not sharp. In these cases, the rank of \(S_m\) exceeds the rank of sharp polynomials of that degree by either one or two.

The polynomials \(S_m\) possess other interesting properties. Their coefficients are integers and can be expressed simply in terms of binomial coefficients. One of the most interesting properties is number-theoretic in nature.
Proposition 1. Let $\mathbb{Z}_m [x, y, z]$ be the polynomial ring in three variables over $\mathbb{Z}_m$, the ring of integers modulo $m$. Then $S_m(x, y, z) = x^m + y^m + z^m$ in $\mathbb{Z}_m [x, y, z]$ if and only if $m$ is prime.

In other words, with the exception of the coefficients of the pure terms, all coefficients of $S_m$ are divisible by $m$ if and only if $m$ is prime. We are led to this proposition...
partly by examining the polynomials $S_m$ listed above. We have, however, a deeper reason to suspect that this proposition is true; the family $\{F_m\}$ of sharp polynomials mentioned above satisfies an analogue of Proposition 1. We will establish an explicit relationship between our family $\{S_m\}$ of extreme symmetric polynomials and the family $\{F_m\}$ of sharp polynomials.

Before we can explore this connection, we must define $S_m$ for general $m$. Let $r = \lfloor \frac{m}{2} \rfloor$. Set $K_{m,0} = 1$, and set

$$K_{m,k} = \binom{m-k}{k} + \binom{m-1-k}{k-1} \quad \text{for} \quad 1 \leq k \leq r. \quad (1)$$

The form of $S_m$ depends on the congruence class modulo 6 of the degree $m$. Suppose first that $m \equiv 1 \mod 6$ or $m \equiv 5 \mod 6$. In the first case, we may write $m = 6L + 1$ for some nonnegative integer $L$ and in the second case we may write $m = 6L - 1$ for some positive integer $L$. In either case, set

$$S_m = x^m + y^m + z^m + \sum_{j=0}^{L-1} (-1)^r K_{m,r-j} (xyz)^{2j+1} [ (xy)^{r-3j-1} + (xz)^{r-3j-1} + (yz)^{r-3j-1} ]. \quad (2)$$

If $m \equiv 3 \mod 6$, we write $m = 6L + 3$ and set

$$S_m = \text{right-hand side of (2)} + (-1)^r L K_{m,r-L} (xyz)^{2L+1}. \quad (3)$$

Similarly, if $m \equiv 4 \mod 6$, we write $m = 6L - 2$, and if $m \equiv 2 \mod 6$, we write $m = 6L - 4$. In either case, we set

$$S_m = x^m + y^m + z^m - \sum_{j=0}^{L-1} (-1)^r K_{m,r-j} (xyz)^{2j} [ (xy)^{r-3j} + (xz)^{r-3j} + (yz)^{r-3j} ]. \quad (4)$$

Finally, if $m \equiv 0 \mod 6$, we write $m = 6L$ and set

$$S_m = \text{right-hand side of (4)} - (-1)^r L K_{m,r-L} (xyz)^{2L}. \quad (5)$$

2. CONNECTION WITH SEVERAL COMPLEX VARIABLES. Because Questions 1 and 2 seem purely algebraic, it is surprising that the sharp polynomials $\{F_m\}$ were discovered in the course of exploring an important and long-standing open problem in the theory of functions of several complex variables. In this section, we describe this connection and state the properties of sharp polynomials. J. D’Angelo is primarily credited with the discovery of these polynomials and their properties (see [1] and [2]).

A continuous mapping $f : X \to Y$ between topological spaces is proper if $f^{-1}(K)$ is compact in $X$ whenever $K$ is compact in $Y$. An important problem is to understand proper holomorphic mappings between balls in complex Euclidean spaces of different dimensions. We focus here on proper polynomial mappings. Let $B_n$ denote the unit ball in $\mathbb{C}^n$ and consider a polynomial mapping $p = (p_1, \ldots, p_N) : \mathbb{C}^n \to \mathbb{C}^N$. Then $p$ maps $B_n$ to $B_N$ properly if and only if it maps the unit sphere in $\mathbb{C}^n$ to the unit sphere in $\mathbb{C}^N$. 

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\[ \|p(z)\|^2 = \sum_{k=1}^{N} |p_k(z)|^2 = 1 \quad \text{whenever} \quad \|z\|^2 = \sum_{j=1}^{n} |z_j|^2 = 1. \quad (6) \]

**Example 1.** Consider the mapping from \( \mathbb{C}^2 \) to \( \mathbb{C}^3 \) given by
\[(z, w) \mapsto (z^3, \sqrt{3}zw, w^3).\]
If \(|z|^2 + |w|^2 = 1\), then
\[
|z^3|^2 + |\sqrt{3}zw|^2 + |w^3|^2 = |z^3|^2 + 3|zw|^2(|z|^2 + |w|^2) + |w^3|^2
= |z|^6 + 3|z|^4|w|^2 + 3|z|^2|w|^4 + |w|^6
= (|z|^2 + |w|^2)^3 = 1,
\]
and thus this mapping is a proper polynomial mapping from \( B_2 \) to \( B_3 \).

We further focus on the case \( n = 2 \). The complexity of a proper polynomial mapping \( p: B_2 \to B_N \) is connected to the target dimension \( N \); a conjecture of D’Angelo \([2]\) states that the degree \( m \) of such a proper mapping must satisfy
\[ m \leq 2N - 3. \quad (7) \]
D’Angelo, Kos, and Riehl \([3]\) prove this inequality in the case in which all components of \( p \) are monomials and provide a family of monomial mappings for which \( m = 2N - 3 \). For each odd \( m \), the squared norm of their sharp mapping is
\[ \|p(z, w)\|^2 = (|z|^2)^m + (|w|^2)^m + \sum_{k=1}^{(m-1)/2} K_{m,k} (|z|^2)^{m-2k} (|w|^2)^k, \quad (8) \]
where the coefficients \( K_{m,k} \) are the same as those defined in (1). Replacing \((|z|^2, |w|^2)\) with \((x, y)\) gives a polynomial in two real variables with nonnegative coefficients, with value 1 on the line \( x + y = 1 \), and with rank \( N = \frac{m+3}{2} \). Clearly there is a bijection between the class of proper monomial mappings from \( B_2 \) to the unit ball in some \( \mathbb{C}^N \) and the class \( \mathcal{P} \) of polynomials in two variables with nonnegative coefficients taking value 1 when \( x + y = 1 \).

We modify (8) to define a family \( \{f_m\} \) of polynomials of both even and odd degrees, although only those of odd degree are in class \( \mathcal{P} \). We set
\[ f_m(x, y) = x^m - (-y)^m + \sum_{k=1}^{[m/2]} K_{m,k} x^{m-2k} y^k. \quad (9) \]
Note that each \( f_m \) is invariant under the map \((x, y) \mapsto (\eta x, \eta^2 y)\) for \( \eta \) an \( m \)th root of unity. Our goal is to use the family \( \{f_m\} \) to define a family \( \{F_m\} \) of sharp polynomials.

Lebl and Peters \([5]\) have made some important contributions to the study of proper monomial mappings between balls. They show that when \( n = 2 \), the degree estimate (7) holds for a class of polynomials with class \( \mathcal{P} \) as a proper subset. They begin by considering a *projectivized* version of the problem; for each polynomial \( p \in \mathcal{P} \) of degree...
m, we homogenize \( p(x, y) - 1 \) with \( z \) and then replace \( z \) with \(-z\) to obtain a homogeneous polynomial \( P(x, y, z) \) of degree \( m \) such that \( P(x, y, z) = q(x, y, z)(x + y + z) \) for a quotient polynomial \( q \) that is homogeneous of degree \( m - 1 \). The condition that \( p \) has nonnegative coefficients becomes a rather awkward condition on the signs of the coefficients of \( P \). Lebl and Peters show, however, that the degree estimate \( R(P) \geq m + 5 \) can be proved under weaker hypotheses than that \( P \) arises from an element of \( \mathcal{P} \). Their first hypothesis says simply that the terms of \( P \) have no common monomial factor of positive degree. Their second hypothesis involves the Newton diagram of the quotient \( q \). They prove the following theorem.

**Theorem 2 (Lebl and Peters [5]).** Let \( P(x, y, z) \) be a homogeneous polynomial of degree \( m \) such that \( P(x, y, z) = q(x, y, z)(x + y + z) \) for a homogeneous polynomial \( q \) of degree \( m - 1 \). Suppose that the terms of \( P \) have no common monomial factor of positive degree and that the Newton diagram of \( q \) is a connected graph. Then

\[
R(P) \geq \frac{m + 5}{2},
\]

and the inequality is sharp.

We obtain a family \( \{F_m\} \) for which equality holds by homogenizing \( f_m(x, y) - 1 \) with \( z \) and then replacing \( z \) with \(-z\), as above:

\[
F_m(x, y, z) = x^m - (-y)^m - (-z)^m + \sum_{k=1}^{\left\lceil \frac{m}{2} \right\rceil} (-1)^k K_{m,k} x^m y^{m-2k} z^k.
\]

The polynomials \( f_m \) and \( F_m \) have many interesting properties. We collect those we will use in the following proposition.

**Proposition 2 (Properties of \( f_m \) and \( F_m \)).** Let \( f_m \) and \( F_m \) be defined as in (9) and (10). Thus \( F_m(x, y, -1) + 1 = f_m(x, y) \).

1. \( f_m(x, y) \) is the unique polynomial satisfying the following:
   (a) \( f_m(0, 0) = 0 \);
   (b) \( f_m(x, y) = 1 \) when \( x + y = 1 \);
   (c) \( f_m \) has degree \( m \);
   (d) \( f_m(\eta x, \eta^2 y) = f_m(x, y) \) for \( \eta \) a primitive \( m \)th root of unity.
2. \( F_m(x, y, z) \) is divisible by \( x + y + z \), and the quotient is a full polynomial.
3. If \( m \) is odd, \( R(F_m) = \frac{m + 5}{2} \). Thus \( F_m \) is a sharp polynomial.
4. The coefficients \( K_{m,k} \) are integers. If \( m \) is prime, for \( 1 \leq k \leq \left\lfloor \frac{m}{2} \right\rfloor \), \( K_{m,k} \equiv 0 \) mod \( m \), whereas if \( m \) is not prime, there exists a \( k \geq 1 \) for which \( K_{m,k} \) is not congruent to \( 0 \) modulo \( m \). That is, \( F_m(x, y, z) = x^m + y^m + z^m \) in \( \mathbb{Z}_m[x, y, z] \) if and only if \( m \) is prime.

**Proof.** Part (1) is the special case of Proposition 2.5 in [1] with \( q = 2 \). The first statement in (2) simply restates property (b) in (1). Part (3) follows from the results proved in [3] since \( F_m(x, y, z) \) is obtained by homogenizing \( f_m(x, y) - 1 \) and \( R(f_m) = \frac{m + 3}{2} \). Part (4) is essentially Corollary 2.8 in [1].
It thus remains only to prove that $F_m$ has full quotient. Although this fact about $F_m$ seems to be known, no proof appears in the literature. We claim that the quotient is

$$Q_m = \sum_{j=1}^{m-1} (-1)^j \sum_{k=0}^{\min[m-1-j, j-1]} \binom{m-1-j}{k} x^{m-1-k-j} [z^j y^k + y^j z^k]$$

$$+ \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^j \binom{m-1-j}{j} x^{m-1-2j} y^j z^j,$$

where, by convention, $\binom{0}{0} = 0$. One must verify that $Q_m(x, y, z)(x + y + z)$ indeed equals $F_m(x, y, z)$. This argument is similar to the argument below that the $S_m$ have full quotient and we omit it.

Our proof of Theorem 1 will show that, if $m \equiv 1, 3 \pmod{6}$, then $R(S_m) = \frac{m+5}{2}$. Therefore it follows from Proposition 2 that, for such $m$, both $F_m$ and $S_m$ are sharp polynomials of degree $m$. Thus, in general, there is not a unique sharp polynomial of each degree. Ours is not the first result bearing on the question of uniqueness; D’Angelo and Lebl [4] address the question of whether $f_m(x, y)$ is the unique polynomial in two variables of degree $m$ in class $\mathcal{P}$ satisfying (7). This question is equivalent to asking whether, for each degree $m$, $F_m(x, y, z)$ is the unique polynomial divisible by $x + y + z$ with rank $\frac{m+5}{2}$ for which $F_m(x, y, -1) + 1$ has positive coefficients. Even with this additional restriction, they prove that, for infinitely many $m$, uniqueness fails. (They do not, however, address the question of whether other sharp polynomials have full quotient.)

D’Angelo and Lebl give a concrete method for constructing new sharp examples from the $f_m$ by replacing expressions in $f_m$ with expressions congruent modulo $x + y - 1$. We use a similar process to go from $F_m$ to $S_m$. A key congruence that we will use repeatedly comes immediately from the statement that $F_m(x, y, z) \equiv 0 \pmod{(x + y + z)}$. Suppose $r = \lfloor \frac{m}{2} \rfloor$. Then

$$G_m(x, y, z) := x^m + \sum_{k=1}^{r} (-1)^k K_{m,k} x^{m-2k} y^k z^k \equiv (-1)^m [y^m + z^m] \pmod{(x + y + z)}.$$  

Proposition 3 describes in all cases how to use the fundamental congruence (12) to obtain $S_m$ from $F_m$. Because it is nonobvious and notation intensive, we illustrate the process with an example.

**Example 2.** Take $m = 13$. Then

$$F_{13} = x^{13} + y^{13} + z^{13} - 13x^{11}yz + 65x^9y^2z^2 - 156x^7y^3z^3 + 182x^5y^4z^4 - 91x^3y^5z^5 + 13xy^6z^6.$$  

Observe that

$$156x^7y^3z^3 = 65x^7y^3z^3 + 91x^7y^3z^3.$$  

Making this substitution, reordering terms, and factoring gives

$$F_{13} = x^{13} + y^{13} + z^{13} - 13x^{11}yz + 65x^9y^2z^2 - 65x^7y^3z^3 + 13xy^6z^6 - 91x^7y^3z^3 + 182x^5y^4z^4 - 91x^3y^5z^5.$$  

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\[\begin{align*}
&= x^{13} + y^{13} + z^{13} + 13xyz[x^5(-x^5 + 5x^3yz - 5xy^2z^2) + y^5z^5] \\
&\quad - 91x^3y^3z^3[x^2(x^2 - 2yz) + y^2z^2].
\end{align*}\]

We recognize that \(-x^5 + 5x^3yz - 5xy^2z^2 = -G_5\) and that \(x^2 - 2yz = G_2\). We may therefore use the fundamental congruence (12) to obtain

\[\begin{align*}
F_{13} &= x^{13} + y^{13} + z^{13} + 13xyz[x^5(G_5) + y^5z^5] - 91x^3y^3z^3[x^2(G_2) + y^2z^2] \\
&\equiv x^{13} + y^{13} + z^{13} + 13xyz[y^5(z^5) + y^5z^5] \\
&\quad - 91x^3y^3z^3[x^2(y^2 + z^2) + y^2z^2] \mod (x + y + z) \\
&= S_{13}.
\end{align*}\]

**Proposition 3.** Let \(G_m = F_m + (-y)^m + (-z)^m\) for \(m \geq 1\) and let \(r = \left\lfloor \frac{m}{2} \right\rfloor\).

1. If \(m = 6L + 1\) or \(m = 6L - 1\),

\[
F_m = x^m + y^m + z^m \\
+ \sum_{j=0}^{L-1} (-1)^{r-j} K_m, r-j \cdot (xyz)^{2j+1} [(-x)^{r-3j-1} G_{r-3j-1} + (yz)^{r-3j-1}],
\]

whereas if \(m = 6L + 3\),

\[
F_m = \text{right-hand side of (13)} + (-1)^{r-L} K_m, r-L \cdot (xyz)^{2L+1}.
\]

We obtain \(S_m\) by replacing \((-1)^{r-3j-1} G_{r-3j-1}\) with the expression \(y^{r-3j-1} + z^{r-3j-1}\), which is congruent to it modulo \(x + y + z\).

2. If \(m = 6L - 2\) or \(m = 6L - 4\),

\[
F_m = -x^m - y^m - z^m \\
+ \sum_{j=0}^{L-1} (-1)^{r-j} K_m, r-j \cdot (xyz)^{2j} [(-x)^{r-3j} G_{r-3j} + (yz)^{r-3j}],
\]

whereas if \(m = 6L\),

\[
F_m = \text{right-hand side of (15)} + (-1)^{r-L} K_m, r-L \cdot (xyz)^{2L}.
\]

We obtain \(-S_m\) by replacing \((-1)^{r-3j} G_{r-3j}\) with the expression \(y^{r-3j} + z^{r-3j}\), which is congruent to it modulo \(x + y + z\).

We needn’t do any tedious algebra to prove the proposition; instead, we use the characterization of \(f_m\) given in part (1) of Proposition 2.

**Proof.** Suppose \(m\) is odd. The argument for even \(m\) is almost identical and is omitted. Replacing \((-1)^{r-3j-1} G_{r-3j-1}\) with the expression \(y^{r-3j-1} + z^{r-3j-1}\) clearly gives the formula for \(S_m\). Thus by the fundamental congruence (12), the expression on the right-hand side of each of (13) and (14) above is congruent to \(S_m\) modulo \(x + y + z\). Consequently, each vanishes when \(x + y + z = 0\). Hence if we replace \(z\) with \(-1\) in
each and add the constant 1, we obtain a polynomial that takes the value 1 on the line $x + y = 1$. The resulting polynomial also clearly vanishes at $(0, 0)$ and is invariant under $(x, y) \mapsto (\eta x, \eta^2 y)$ for $\eta$ a primitive $m$th root of unity. Thus it is equal to $f_m(x, y)$ by Proposition 2, part (1). Subtracting 1, homogenizing, and replacing $z$ with $-z$ gives the result.

3. PROOF OF THE MAIN THEOREM. It is apparent from the explicit formulas (2) and (3) that if $m \equiv 1, 3 \mod 6$, then $R(S_m) = \frac{m+5}{2}$. Furthermore, since $S_m$ includes the monomials $x^m$, $y^m$, and $z^m$, the terms of $S_m$ have no common monomial factor. Because a full polynomial has a connected Newton diagram, if we can show that $S_m = sq_m$ for a full polynomial $q_m$, then it will follow from the theorem of Lebl and Peters (Theorem 2) that when $m \equiv 1, 3 \mod 6$, $S_m$ is a sharp polynomial. We define the quotient polynomial $q_m$ explicitly, so that the remainder of the proof is a simple verification that $S_m$ is the resulting product polynomial.

**Proof.** Suppose first that $m = 2r + 1$. Let $q_{2r+1}(x, y, z) = \sum \gamma(a, b, c)x^ay^bz^c$, where

$$\gamma(2r - j, k, j - k) = (-1)^j \binom{j}{k}$$

for $2r - j \geq k \geq j - k \geq 0$, with the same value for $\gamma(\sigma(2r - j, k, j - k))$ for any other permutation $\sigma(2r - j, k, j - k)$ of these numbers. As with the polynomials $S_m$, the structure of the quotient polynomials is perhaps better seen by looking at the Newton diagram than by looking at the formula. Figure 6 shows the Newton diagram for $q_7$.

![Figure 6. Newton diagram for $q_7$.](image-url)
The proof is somewhat tedious and consists of demonstrating that the coefficients of \( q_{2r+1}s \) agree with those of \( S_{2r+1} \). Let \( \alpha(A, B, C) \) be the coefficient of \( x^A y^B z^C \) in \( q_{2r+1}(x, y, z) s(x, y, z) \). Then \( A + B + C = 2r + 1 \) and

\[
\alpha(A, B, C) = \gamma(A - 1, B, C) + \gamma(A, B - 1, C) + \gamma(A, B, C - 1),
\]

where we take \( \gamma(a, b, c) = 0 \) if one of the arguments is negative. We must show that the coefficients \( \alpha(A, B, C) \) agree with the coefficients of the polynomial \( S_{2r+1} \). In principle, there are many cases to consider. We note, however, that \( q_{2r+1} \) and \( q_{2r+1}s \) are symmetric. Thus we need only consider \( \alpha(A, B, C) \) for \( A \geq B \geq C \).

Case 1. If \( B = C = 0 \), then

\[
\alpha(2r + 1, 0, 0) = \gamma(2r, 0, 0) = 1,
\]

which agrees with the coefficient of \( x^{2r+1} \) in \( S_{2r+1} \).

Case 2. Suppose next that \( A \geq B > C = 0 \). Observe that, because \( A + B = 2r + 1 \), we must in fact have \( A > B \). Thus

\[
\alpha(A, B, 0) = \gamma(A - 1, B, 0) + \gamma(A, B - 1, 0) = (-1)^B + (-1)^{B-1} = 0.
\]

We have now dealt fully with the cases in which at least one of \( A, B, \) or \( C \) is zero.

Case 3. Next consider \( A > B \geq C > 0 \). Write \( (A, B, C) = (2r + 1 - j, k, j - k) \) for natural numbers \( j \) and \( k \) with \( j > k \). Thus

\[
\alpha(2r + 1 - j, k, j - k)
\]

\[
= \gamma(2r - j, k, j - k) + \gamma(2r - (j - 1), k - 1, (j - 1) - (k - 1))
\]

\[
+ \gamma(2r - (j - 1), k, (j - 1) - k)
\]

\[
= (-1)^{j-1} \left( \binom{j}{k} + (-1)^{j-1} \binom{j-1}{k-1} \right) + (-1)^{j-1} \binom{j-1}{k} = 0.
\]

Case 4. We next treat the case in which \( A = B > C > 0 \). Write \( (A, B, C) = (2r + 1 - j, 2r + 1 - j, 2j - 2r - 1) \) for some \( j \). Thus

\[
\alpha(2r + 1 - j, 2r + 1 - j, 2j - 2r - 1)
\]

\[
= \gamma(2r - j, 2r - (j - 1), 2(j - 1) - 2r + 1)
\]

\[
+ \gamma(2r - (j - 1), 2r - j, 2(j - 1) - 2r + 1)
\]

\[
+ \gamma(2r - (j - 1), 2r + 1 - j, 2(j - 1) - 2r)
\]

\[
= (-1)^{j-1} \left( \binom{j-1}{2r-j} + (-1)^{j-1} \binom{j-1}{2r-j} + (-1)^{j-1} \binom{j-1}{2r+1-j} \right)
\]

\[
= (-1)^{j-1} \left[ \left( \binom{j-1}{2r-j} + \binom{j}{2r+1-j} \right) \right]
\]

\[
= (-1)^{j-1} K_{2r+1, 2r+1-j}.
\]
Case 5. Finally, if $2r + 1 = 6L + 3$, then $A = B = C = 2L + 1$ is possible. In this case,

$$\alpha(2L + 1, 2L + 1, 2L + 1) = \gamma(2L, 2L + 1, 2L + 1) + \gamma(2L + 1, 2L, 2L + 1) + \gamma(2L + 1, 2L + 1, 2L) = 3\gamma(2L + 1, 2L + 1, 2L) = 3\gamma(2r - (4L + 1), 2L + 1, 2L) = 3(-1)^{4L+1}\left(\frac{4L+1}{2L+1}\right)$$

$$= (-1)^{4L+1}\left[\frac{4L+1}{2L+1} + \left(\frac{4L+1}{2L+1}\right) + \left(\frac{4L+1}{2L+1}\right)\right]$$

$$= (-1)^{4L+1}\left[\frac{4L+1}{2L+1} + \frac{4L+2}{2L+1}\right]$$

$$= (-1)^{4L+1}K_{6L+3,2L+1}.$$  

Thus if $r \equiv 0, 2 \mod 3$,

$$q_{2r+1}s = x^{2r+1} + y^{2r+1} + z^{2r+1} + \sum_{j=r+1}^{\lfloor \frac{4r+1}{2} \rfloor} (-1)^{j-1}K_{2r+1,2r+1-j}\left[(xy)^{2r+1-j}z^{2(j-r)-1} + (xz)^{2r+1-j}y^{2(j-r)-1} + x^{2(j-r)-1}(yz)^{2r+1-j}\right]$$

$$= x^{2r+1} + y^{2r+1} + z^{2r+1} + \sum_{j=0}^{\lfloor \frac{4r+1}{2} \rfloor} (-1)^{r+j}K_{2r+1,r-j}\left[(xy)^{r-j}z^{2j+1} + (xz)^{-j}y^{2j+1} + x^{2j+1}(yz)^{-j}\right].$$

If $2r + 1 = 6L + 3$, we get the last expression above together with the additional term $-K_{2r+1,2L+1}(xyz)^{2L+1}$. Thus in either case $q_{2r+1}s = S_{2r+1}$, as claimed.

If $m$ is even, we write $m = 2r$ and let $q_{2r}(x, y, z) = \sum \gamma(a, b, c)x^ay^bz^c$, with

$$\gamma(2r - 1 - j, k, j - k) = (-1)^{j+1}\binom{j}{k}$$

for $2r - 1 - j \geq k \geq j - k \geq 0$, with the same value for $\gamma(\sigma(2r - 1 - j, k, j - k))$ for any other permutation $\sigma(2r - 1 - j, k, j - k)$ of these numbers. The proof that $q_{2r}s = S_{2r}$ is analogous to the above and we omit it.

4. PROOF OF PROPOSITION 1. We turn now to Proposition 1. Since the coefficients of $S_m$ are a subset of the coefficients of $F_m$, one direction follows immediately from part (4) of Proposition 2. We may also give a direct proof of both directions using the idea of D’Angelo’s proof.
Proof. Suppose $m = 6L + 1$ or $m = 6L - 1$. The cases for $m = 6L + 3$ and for $m$ even are analogous. Taking $z = -x - y$ gives

$$S_m(x, y, -(x + y)) + (x + y)^m = x^m + y^m + \sum_{j=0}^{L-1} (-1)^{r+j+1} K_{m,r-j} \cdot \left[ xy(x+y)^{2j+1}(xy)^{r-1-3j} + (-x-y)(x^{r-1-3j} + y^{r-1-3j}) \right]$$

$$= x^m + y^m + \sum_{j=0}^{L-1} (-1)^{r+j+1} K_{m,r-j} q_j(x, y).$$

Because $S_m(x, y, -(x + y)) = 0$, we obtain the identity

$$(x + y)^m = x^m + y^m + \sum_{j=0}^{L-1} (-1)^{r+j+1} K_{m,r-j} q_j(x, y).$$

It is well known that, as elements of the vector space $\mathbb{Z}_m[x, y]$, $(x + y)^m = x^m + y^m$ if and only if $m$ is prime. Therefore the same is true of the equivalent expression $x^m + y^m + \sum_{j=0}^{L-1} (-1)^{r+j+1} K_{m,r-j} q_j(x, y)$. Because the degree of $q_j$ in $x$ is $2(2j+1) + 2(r - 1 - 3j) = 2r - 2j$, the $q_j$ have different degrees in $x$ and are hence linearly independent in $\mathbb{Z}_m[x, y]$. Since $\sum_{j=0}^{L-1} (-1)^{r+j+1} K_{m,r-j} q_j(x, y)$ is the zero polynomial if and only if $m$ is prime, we conclude that if $m$ is prime, then $K_{m,r-j} \equiv 0 \mod m$ for all $j$, whereas if $m$ is not prime, then there exists $j$ for which $K_{m,r-j}$ is not congruent to zero modulo $m$.

**ACKNOWLEDGMENTS.** The author acknowledges support from NSF grant DMS 1200815. The author would also like to thank John D’Angelo for his encouragement.

**REFERENCES**


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