Primitive lattice points in planar domains

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§1 Introduction.

Let $\mathcal{D}$ be a compact convex set in $\mathbb{R}^2$, containing $\mathbf{0}$ as an interior point, having a smooth boundary curve $\mathcal{C}$ with nowhere vanishing curvature. How many *primitive* lattice points (m, n) ($m \in \mathbb{Z}$, $n \in \mathbb{Z}$, m, n coprime) are in $\sqrt{x}\mathcal{D}$ for large x? If we write $\mathcal{A}_{\mathcal{D}}(x)$ for the number of such primitive points, the answer is certainly of the form

$$\mathcal{A}_{\mathcal{D}}(x) = \frac{6}{\pi^2} m(\mathcal{D})x + O(x^{\theta+\epsilon})$$

for every $\epsilon > 0$, for some θ (independent of $\mathcal{D}$) satisfying $\frac{1}{4} \le \theta \le \frac{1}{2}$. (Implied constants may depend on $\mathcal{D}$ and ϵ unless otherwise specified.) Here, of course, $m(\mathcal{D})$ is the area of $\mathcal{D}$.

Moroz [13], Hensley [7], Huxley and Nowak [10], Müller [14] and Zhai [21] have treated this question assuming the Riemann hypothesis (R.H.), with improving values of θ culminating in

$$\theta_Z = \frac{33349}{84040} = 0.3968\ldots$$

(Zhai [21]). The smoothness assumptions on $\mathcal{C}$ vary in these papers. Indeed, Zhai's curves $\mathcal{C}$ are only piecewise smooth, and $\mathcal{D}$ is not necessarily convex; but Zhai imposes Diophantine approximation conditions on the tangent slopes on either side of the 'corners' of $\mathcal{C}$ (if any).

In the present paper, I improve Zhai's constant θ_Z. For simplicity I assume that the tangent slopes on either side of the 'corners' of $\mathcal{C}$ are rational. I shall make the following hypotheses about $\mathcal{D}$, somewhat similar to those in Nowak [15].

(H1) $\mathcal{D}$ is a compact set whose boundary curve $\mathcal{C} = \partial\mathcal{D}$ can be written in polar coordinates as

$$r = \rho(\theta), \quad 0 \le \theta \le 2\pi,$$

where ρ is positive and continuous.

(H2) There is a partition

$$\theta_0 < \theta_1 < \cdots < \theta_J = 2\pi + \theta_0$$

such that $\rho^{(4)}$ is continuous on $[\theta_{j-1}, \theta_j]$ and the curvature of

$$\mathcal{C}_j : r = \rho(\theta), \quad \theta \in [\theta_{j-1}, \theta_j]$$

is never zero. No tangent to $\mathcal{C}_j$ passes through $\mathbf{0}$. No $\mathcal{C}_j$ has both a horizontal and a vertical tangent.

(H3) The reciprocal curves $\mathcal{R}(\mathcal{C}_j)$ $(j = 1, \dots J)$ can be written in the form

$$r = \rho_j(\theta) \quad (\lambda_{j-1} \le \theta \le \lambda_j)$$

with $\rho_j^{(4)}$ continuous on $[\lambda_{j-1}, \lambda_j]$.

(H4) The tangents to $\mathcal{C}_j$ at $\theta = \theta_{j-1}$, $\theta = \theta_j$ have rational slopes.

For more details about reciprocal curves, see Huxley [8], Lemma 4, and for a brief summary, §2 below.

In particular, suppose that $\mathcal{D}$ is convex, (H1) holds with $\rho^{(4)}$ continuous (as a function of period 2π), and $\mathcal{C}$ has nowhere zero curvature. Suppose also that $\mathcal{R}(\mathcal{C})$ is given by

$$r = \rho_1(\theta),$$

$\rho_1^{(4)}$ continuous as a function of period 2π. Then (H2)–(H4) are automatically satisfied. The main point here is that because the tangent to $\mathcal{C}$ varies continuously in slope, we can choose partition points $\theta_0, \dots, \theta_{J-1}$ so that the tangent slope is rational at the corresponding points on $\mathcal{C}$.

Theorem 1 *Assume R. H., and let $\mathcal{D}$ be a compact set with the properties (H1)–(H4). The number of primitive lattice points in $\sqrt{x}\mathcal{D}$ is*

$$\mathcal{A}_{\mathcal{D}}(x) = \frac{6}{\pi^2}\, m(\mathcal{D})x + O(x^{5/13+\epsilon}).$$

For comparison with θ_Z, we note that $5/13 = 0.3846\ldots$. When $\mathcal{D}$ is the unit disc, we have the stronger exponent $221/608+\epsilon = 0.3634\ldots$ (Wu, [20]).

For $\mathcal{D}$ as in Theorem 1, let

$$Q(\mathcal{D}; \boldsymbol{u}) = Q(\boldsymbol{u}) = \inf\{\tau^2 : \boldsymbol{u}/\tau \in \mathcal{D}\}, \tag{1.1}$$

whenever $\boldsymbol{u} \in \mathbb{R}^2$. The **Hlawka zeta function** of $\mathcal{D}$ is the meromorphic function $Z_{\mathcal{D}}(s)$ obtained by extending

$$Z_{\mathcal{D}}(s) = \sum_{\boldsymbol{m}\in\mathbb{Z}^2,\ \boldsymbol{m}\neq 0} Q(\boldsymbol{m})^{-s} \quad (\operatorname{Re} s > 1).$$

It is well-known (and will be shown during calculations in §3) that $Z_{\mathcal{D}}(s)$ is analytic in $\operatorname{Re} s > 1/3$ except for a simple pole with residue $m(\mathcal{D})$ at $s = 1$. With more effort, the domain can be enlarged, but we shall not need this. For results such as Theorem 1, we need to find σ as small as possible such that the bound

$$\int_T^{2T} |Z_{\mathcal{D}}(\sigma + it)|^2 dt \ll T^{1+\epsilon} \quad (T \geq 2) \tag{1.2}$$

holds. The successive conditions on σ published so far that suffice for (1.2) are

$$\sigma \geq 0.75 \text{ (Huxley and Nowak [10]),}$$
$$\sigma \geq 48/73 = 0.6575\ldots \text{ (Müller [14]),}$$
$$\sigma \geq 749/1168 = 0.6412\ldots \text{ (Zhai [21]).}$$

Dr. Müller kindly sent me a sketch of an argument (based on a lecture of Huxley) that gives the value $\sigma = 0.625$. Even though I could not substantiate all the details, this sketch helped me understand the difficulties of this problem.

Theorem 2 *Let $\mathcal{D}$ be a compact set with the properties (H1)–(H4). Then* (1.2) *holds for $\sigma \geq 3/5$.*

The following estimate is an important tool for the proof of Theorem 2. It is of some interest that condition (H4) can be omitted in Theorems 3 and 4.

Theorem 3 *Let $\mathcal{D}$ be a compact set with the properties (H1)–(H3). Let $X > 0$, $\Delta > 0$. The number of solutions $\mathcal{N}_{\mathcal{D}}(X, \Delta)$ of*

$$Q(\boldsymbol{m}) \sim X \ , \ 0 \le Q(\boldsymbol{n}) - Q(\boldsymbol{m}) < \Delta X \tag{1.3}$$

satisfies

$$\mathcal{N}_{\mathcal{D}}(X, \Delta) \ll 1 + X^{6/5+\epsilon} + \Delta X^2.$$

The notation '$\Gamma \sim X$' means $X < \Gamma \le 2X$.

It seems that the best previous result is

$$\mathcal{N}_D(X, \Delta) \ll 1 + X^{547/416+\epsilon} + \Delta X^2,$$

which can be deduced from the work of Huxley [9] by treating $\boldsymbol{m}$ trivially in (1.3).

I conjecture that 6/5 could be replaced by 1 in Theorem 3. A proof of this would give a new approach to the work of Robert and Sargos [16] on the number of solutions $\mathcal{N}$ of

$$|n_1^\beta + n_2^\beta - n_3^\beta - n_4^\beta| < \Delta N^\beta \ , \ n_j \sim N.$$

Here β is real, $\beta \ne 0, 1$. It is perhaps not surprising that this special case has a stronger result attached to it, namely

$$\mathcal{N} \ll N^{2+\epsilon} + \Delta N^{4+\epsilon}.$$

The following result is a step towards Theorem 2, and is almost a corollary of Theorem 3.

Theorem 4 *Make the hypotheses of Theorem 3. Let*

$$S_X(s) = \sum_{\boldsymbol{m} \ne \boldsymbol{0},\ Q(\boldsymbol{m}) \le X} Q(\boldsymbol{m})^{-s}.$$

Then for $3/5 \le \sigma \le 3/4$, $X \ge 1$, $T \ge 2$,

$$X^{2-2\sigma} \le T,$$

we have

$$\int_T^{2T} |S_X(\sigma + it)|^2 dt \ll T^{1+\epsilon}.$$

Theorem 3 will be proved in §2, and Theorem 4 is deduced from it there. This enables us to prove Theorem 2 in §3. In §4, we recall a standard decomposition.

$$A_D(x) - \frac{6}{\pi^2}\, m(D)x = E_1(x) + E_2(x),$$

and prove that $E_2(x) = O(x^{5/13+\epsilon})$ via Perron's formula and Theorem 2. This is where R. H. is needed; the strategy follows Huxley and Nowak [10]. We then show that $E_1(x) = O(x^{5/13+\epsilon})$ via a refinement of exponential sum estimates of Zhai [21]. Theorem 3 is used again in treating $E_1(x)$.

An idea of Montgomery and Vaughan [12] underpins [10, 14, 21] and the present work, although the details of [12] are totally different. The present paper uses some techniques from my paper [3], and I quote one lemma from [3].

I would like to acknowledge the friendly hospitality of the Mathematics Department of the University of Florida, where part of the work was done.

§2 Proof of Theorem 3 and the deduction of Theorem 4.

I begin with an elementary lemma that I have not been able to find in the literature. Let

$$\psi(x) = x - [x] - 1/2,$$

$$\psi_0(x) = \begin{cases} \psi(x) & (x \notin \mathbb{Z}) \\ 0 & (x \in \mathbb{Z}), \end{cases}$$

$$\psi^*(x) = \begin{cases} \psi(x) & (x \notin \mathbb{Z}) \\ 1/2 & (x \in \mathbb{Z}). \end{cases}$$

Thus $2(\psi_0 - \psi)$ is the indicator function of $\mathbb{Z}$. For $\alpha \neq 0$, β real, let

$$\Psi_{(\alpha,\beta)}(M) = \sum_{1 \le n \le \alpha M} \psi_0\left(\frac{\beta n}{\alpha}\right).$$

Lemma 1 *Let $\alpha > 0$, $\beta > 0$, $M > \max(\alpha^{-1}, \beta^{-1})$. Then*

$$\Psi_{(\alpha,\beta)}(M) + \Psi_{(\beta,\alpha)}(M) = \frac{\beta}{2\alpha}\,\psi^2(\alpha M) + \frac{\alpha}{2\beta}\,\psi^2(\beta M) - \psi(\alpha M)\psi(\beta M) - \frac{\alpha}{8\beta} - \frac{\beta}{8\alpha} + \frac{1}{4}.$$

It is asserted by Nowak [15] that

$$\Psi_{(\alpha,\beta)}(M) = \Psi_{(\beta,\alpha)}(M) + O(1) \tag{2.1}$$

with implied constants depending on α, β. Lemma 1 corrects this to

$$\Psi_{(\alpha,\beta)}(M) = -\Psi_{(\beta,\alpha)}(M) + O(1). \tag{2.2}$$

Proof of Lemma 1. The number of lattice points in the rectangle $[1, \alpha M] \times [1, \beta M]$ is

$$[\alpha M]\,[\beta M] = \alpha\beta M^2 - \alpha M\psi(\beta M) - \beta M\psi(\alpha M) - (\alpha+\beta)\,\frac{M}{2} + \frac{\psi(\alpha M)}{2} + \frac{\psi(\beta M)}{2} + \psi(\alpha M)\psi(\beta M) + 1/4. \tag{2.3}$$

We count these points in another way. The number of them in the triangle $1 \le x \le \alpha M$, $1 \le y \le \beta x/\alpha$, with weight $\frac{1}{2}$ attached to those on the upper edge, is

$$\sum_{1 \le n \le \alpha M} \left\{ \left[\frac{\beta n}{\alpha}\right] - (\psi_0 - \psi)\left(\frac{\beta n}{\alpha}\right) \right\} = \sum_{1 \le n \le \alpha M} \left(\frac{\beta n}{2} - \frac{1}{2}\right) - \Psi_{(\alpha,\beta)}(M)$$

$$= \frac{\beta}{2\alpha}\,(\alpha M - \psi(\alpha M) - 1/2)(\alpha M - \psi(\alpha M) + 1/2)$$

$$-\frac{1}{2}\,(\alpha M - \psi(\alpha M) - 1/2) - \Psi_{(\alpha,\beta)}(M)$$

$$= \frac{\alpha\beta M^2}{2} - \beta M\psi(\alpha M) + \frac{\beta}{2\alpha}\,\psi^2(\alpha M) - \frac{\beta}{8\alpha} - \frac{\alpha M}{2}$$

$$+\frac{\psi(\alpha M)}{2} + \frac{1}{4} - \Psi_{(\alpha,\beta)}(M).$$

Adding this to the corresponding expression with (α, β) interchanged, we find that the right-hand side of (2.3) is equal to

$$\alpha\beta M^2 - \alpha M\psi(\beta M) - \beta M\psi(\alpha M) + \frac{\alpha}{2\beta}\,\psi^2(\beta M) + \frac{\beta}{2\alpha}\,\psi^2(\alpha M)$$

$$-\frac{\alpha}{8\beta} - \frac{\beta}{8\alpha} - (\alpha+\beta)\,\frac{M}{2} + \frac{\psi(\alpha M)}{2} + \frac{\psi(\beta M)}{2} + \frac{1}{2}$$

$$-\Psi_{(\alpha,\beta)}(M) - \Psi_{(\beta,\alpha)}(M).$$

The lemma follows at once.

For $f : I = [a, b] \to \mathbb{R}$ with continuous nowhere vanishing second derivative, we write $g = f$ if $f'' < 0$, $g = -f$ if $f'' < 0$. Let ϕ be the inverse function of g' and write

$$G(u, v) = G(f; u, v) = vg\left(\phi\left(-\frac{u}{v}\right)\right) + u\phi\left(-\frac{u}{v}\right)$$

for $(u, v) \in E(f)$, where

$$E(f) = \{(u, v) : v > 0,\ -vg'(a) \le u \le -vg'(b)\}.$$

We also write, for $M \ge 1$,

$$S(f, M) = \sum_{nM^{-1}\in I} \psi\left(Mf\left(\frac{n}{M}\right)\right),$$

$$S^*(f, M) = \sum_{nM^{-1}\in I} \psi^*\left(Mf\left(\frac{n}{M}\right)\right),$$

and for integer h,

$$S_h(f, M) = \sum_{nM^{-1}\in I} e\left(hMf\left(\frac{n}{M}\right)\right).$$

As usual, $e(\theta)$ denotes $e^{2\pi i\theta}$.

For a compact set $\mathcal{D}$ satisfying (H1)–(H3), we write

$$N_{\mathcal{D}}(x) = \sum_{(m,n)\in x^{1/2}\mathcal{D}} 1,$$

$$N^*_{\mathcal{D}}(x) = \sum_{(m,n)\in x^{1/2}(\mathcal{D}\setminus\mathcal{C})} 1.$$

For $x > 1$, let

$$P_{\mathcal{D}}(x) = N_{\mathcal{D}}(x) - m(\mathcal{D})x, \ P^*_{\mathcal{D}}(x) = N^*_{\mathcal{D}}(x) - m(\mathcal{D})x.$$

Lemma 2 *Let $\mathcal{D}$ be a compact set satisfying (H1)–(H3). We may write $P_{\mathcal{D}}(x)$, $P^*_{\mathcal{D}}(x)$ in the form*

$$P_{\mathcal{D}}(x) = \sum_{j=1}^{J} e_j S(f_j, \sqrt{x}) + O(1),$$

$$P^*_{\mathcal{D}}(x) = \sum_{j=1}^{J} e_j S^*(f_j, \sqrt{x}) + O(1),$$

where $J = O(1)$, $e_j \in \{-1, 1\}$ and $f_j : I_j = [a_j, b_j] \to \mathbb{R}$, $b_j > a_j \geq 0$ has f_j'' nowhere vanishing and $f_j^{(4)}$ continuous. For each j, one of $\mathcal{C}_j^{(1)} = \{(x, f_j(x)) : x \in I_j\}$ or $\mathcal{C}_j^{(2)} = \{(f_j(y), y) : y \in I_j\}$ is $\mathcal{C}_j$.

Proof. This is given by Nowak [15], proof of Corollary 1. The formula (2.1) is used there to interchange the role of the variables in counting lattice points of $\mathcal{D}$ within a sector

$$x > 0, \ y > 0, \quad \frac{\beta_1}{\alpha_1} < \frac{y}{x} < \frac{\beta_2}{\alpha_2}.$$

The reader may verify that in this part of the argument, (2.1) should be replaced by (2.2).

Lemma 3 (Reciprocation). *Let $\mathcal{C}_0$ be an arc in $\mathbb{R}^2$ given in polar coordinates by*

$$r = \rho(\theta), \ a \leq \theta \leq b, \ [a, b] \subset [0, 2\pi],$$

where ρ is positive and $\rho^{(4)}$ is continuous. Suppose that the curvature of $\mathcal{C}_0$ is nowhere 0, and no tangent to $\mathcal{C}_0$ passes through $\mathbf{0}$. Let $(\alpha(\theta),\ \beta(\theta))$ be the point such that the tangent to $\mathcal{C}_0$ at the point $P(\theta)$ with polar coordinates θ, $\rho(\theta)$ has equation

$$\alpha(\theta)x + \beta(\theta)y = 1.$$

We write

$$\mathcal{R}(P(\theta)) = (\alpha(\theta),\ \beta(\theta)).$$

Then $\mathcal{R}(\mathcal{C}_0) = \{(\alpha(\theta),\ \beta(\theta)) : a \le \theta \le b\}$ *is a curve of nowhere zero curvature which can be written in polar coordinates* (R, ϕ) *as*

$$R = \rho_1(\phi) \quad (c \le \phi \le d). \tag{2.4}$$

No tangent to $\mathcal{R}(\mathcal{C}_0)$ *passes through the origin.*

Proof. This is a variant of Lemma 4 of Huxley [8], where $a = 0$, $b = 2\pi$. Most of the proof needs no change. We have

$$\alpha(\theta) = \frac{\sin(\theta+\lambda)}{\rho(\theta)\sin\lambda}\ ,\ \beta(\theta) = -\frac{\cos(\theta+\lambda)}{\rho(\theta)\sin\lambda}\,, \tag{2.5}$$

where λ is the angle between the radius vector from $\mathbf{0}$ to $P(\theta)$ and the tangent at θ; by hypothesis, $\lambda \ne 0$. The radii of curvature $\sigma_1(\theta)$ of $\mathcal{C}_0$ and $\sigma_2(\theta)$ of $\mathcal{R}(\mathcal{C}_0)$ are related by

$$\sigma_1(\theta)\sigma_2(\theta)\sin^3\lambda(\theta) = 1,$$

and this shows that the curvature of $\mathcal{R}(\mathcal{C}_0)$ is nowhere 0. The 'self-reciprocal' property is that the tangent to $\mathcal{R}(\mathcal{C}_0)$ at $(\alpha(\theta),\ \beta(\theta))$ has equation

$$Ax + By = 1,$$

where $(A, B) = P(\theta)$. This tangent does not pass through $\mathbf{0}$.

The representation (2.4) simply requires that a half-line with initial point at $\mathbf{0}$ never intersects $\mathcal{R}(\mathcal{C}_0)$ more than once. If there is such a double intersection, it is an easy exercise in the intermediate value theorem to show that another such half-line is tangent to $\mathcal{R}(\mathcal{C}_0)$, which is absurd.

Lemma 4 *Define* $G(u, v) = G(f; u, v)$ *as above and let*

$$\mathcal{C}_0 = \{(x, g(x)) : a \le x \le b\}.$$

Suppose that $\mathcal{C}_0$ *satisfies the hypotheses of Lemma 3. Then* $G(u, v)$ *is homogeneous of order 1 on* $E(f)$ *with constant sign, say* e^*. *There are positive constants* c_1, c_2 *such that*

$$c_1\sqrt{u^2+v^2} \le |G(u, v)| \le c_2\sqrt{u^2+v^2} \ \text{ on } E(f). \tag{2.6}$$

The set of (u, v) *in* $E(f)$ *satisfying*

$$|G(u, v)| = 1$$

is the curve $e^*\mathcal{R}(\mathcal{C}_0)$.

Proof. This is a variant of material in Nowak [15]. It is clear that G is homogeneous of degree 1 on $E(f)$. For $(u, v) \in E(f)$, there is a unique ζ with

$$a \le \zeta \le b,\ g'(\zeta) = -\frac{u}{v}\,,$$

so that

$$\phi\left(-\frac{u}{v}\right) = \zeta.$$

Conversely each ζ in I corresponds to a half-line of (u, v) in $E(f)$ with $g'(\zeta) = -u/v$. The equation of the tangent to $\mathcal{C}_0$ at $(\zeta, g(\zeta))$ is

$$\begin{aligned} y - g'(\zeta)x &= g(\zeta) - g'(\zeta)\zeta \\ &= g\left(\phi\left(-\frac{u}{v}\right)\right) + \frac{u}{v}\,\phi\left(-\frac{u}{v}\right), \end{aligned}$$

that is

$$ux + vy = G(u, v). \tag{2.7}$$

Since this tangent does not pass through $\mathbf{0}$, we obtain (2.6) for $u^2 + v^2 = 1$ by continuity of G, and the general case by homogeneity. By continuity, G takes only one sign e^* on $E(f)$.

Suppose first that $e^* = 1$. Let $\zeta \in [a, b]$. The point (u, v) in $E(f)$ with $-u/v = g'(\zeta)$ and

$$G(u, v) = 1 \tag{2.8}$$

is clearly $\mathcal{R}(\zeta, g(\zeta))$, and as ζ varies over I, $\mathcal{R}(\zeta, g(\zeta))$ varies over the curve

$$G(u, v) = 1,\ v > 0,\ -vg'(a) \le u \le -vg'(b)$$

as claimed.

Now suppose that $e^* = -1$. The above argument goes through with slight changes; we have

$$|G(u, v)| = 1$$

at the point $-\mathcal{R}(\zeta, g(\zeta))$.

The relevance of $G(f; u, v)$ to Theorem 2 will be seen below when the van der Corput B-process is applied to exponential sums arising from Lemma 2.

Lemma 5 *Let $\mathcal{C}_0$ be as in Lemma 4. There is a compact set $\mathcal{D}$ satisfying (H1)–(H4) such that $\mathcal{C}_0$ is one of the arcs $\mathcal{C}_1, \ldots, \mathcal{C}_J$ of $\mathcal{C} = \partial\mathcal{D}$.*

Proof. We can extend g to an interval $[a-\eta, b+\eta]$, with $\eta > 0$, so that $g^{(4)}$ is continuous and $g'' \neq 0$ on $[a-\eta, b+\eta]$, and no tangent to the curve

$$Q_0 = \{(x, g(x)) : x \in [a-\eta, b+\eta]\}$$

passes through $\mathbf{0}$. We can arrange that $g'(a-\eta)$, $g'(b+\eta)$ are rational by reducing η.

We can now readily construct four circular arcs $Q_1, \ldots, Q_4$ such that

(i) $Q_0, Q_1, \ldots, Q_4$ are nonoverlapping and together form a simple closed curve $\mathcal{C}$ that encloses $\mathbf{0}$;

(ii) the tangents at the endpoints of $Q_1, \ldots, Q_4$ have rational slope.

The compact set $\mathcal{D}$ whose boundary is $\mathcal{C}$ has the required properties.

Lemma 6 *Let $\mathcal{E}_1$, $\mathcal{E}_2$ be finite sets in $\mathbb{Z}^2$ and $F_j : \mathcal{E}_j \to \mathbb{R}$. Let*

$$S_j(\alpha) = \sum_{(h,\ell)\in\mathcal{E}_j} e(F_j(h,\ell)\alpha).$$

Let $\delta > 0$. The number of solutions $\mathcal{N} = \mathcal{N}(F_1, F_2, \delta)$ of

$$|F_1(h_1,\ell_1) - F_2(h_2,\ell_2)| < \delta, \quad (h_j, \ell_j) \in \mathcal{E}_j \tag{2.9}$$

satisfies

$$\mathcal{N} \ll \delta \int_{-1/2\delta}^{1/2\delta} |S_1(\alpha)S_2(\alpha)| d\alpha.$$

If $\mathcal{E}_1 = \mathcal{E}_2$, $F_1 = F_2$, then

$$\mathcal{N} \gg \delta \int_{-1/2\delta}^{1/2\delta} |S_1(\alpha)|^2 d\alpha.$$

The implied constants are absolute.

Proof. This is a variant of Lemma 2.1 of Watt [19]. Let

$$\operatorname{sinc} a = \frac{\sin \pi a}{\pi a} \ (a \in \mathbb{R},\ a \neq 0),\ \operatorname{sinc} 0 = 1,$$

$$\Lambda(a) = \max(0, 1 - |a|).$$

Then

$$\text{sinc}^2 a = \int_{-\infty}^{\infty} \Lambda(b) e(ab) db, \ \Lambda(b) = \int_{-\infty}^{\infty} \text{sinc}^2 b \ e(ab) db.$$

Now $\text{sinc}^2 a \geq 0$ and, for $|a| \leq 1/2$, $\text{sinc}^2 a \geq 4/\pi^2$. Hence

$$\begin{aligned}
\mathcal{N} &\leq \sum_{(h_1,\ell_1)\in\mathcal{E}_1} \sum_{(h_2,\ell_2)\in\mathcal{E}_2} \frac{\pi^2}{4} \text{sinc}^2 \left(\frac{1}{2\delta} \left(F_1(h_1,\ell_1) - F_2(h_2,\ell_2) \right) \right) \\
&= \frac{\pi^2}{2} \delta \int_{-\infty}^{\infty} \Lambda(2\delta\alpha) S_1(\alpha) \overline{S_2(\alpha)} \, d\alpha
\end{aligned}$$

(after an interchange of summation and integration, and a change of variable). Clearly

$$\mathcal{N} \leq \frac{\pi^2}{2} \delta \int_{-1/2\delta}^{1/2\delta} |S_1(\alpha) S_2(\alpha)| d\alpha.$$

Suppose now that $\mathcal{E}_1 = \mathcal{E}_2$, $F_1 = F_2$. Then

$$\begin{aligned}
\mathcal{N} &\geq \sum_{(h_1,\ell_1)\in\mathcal{E}_1} \sum_{(h_2,\ell_2)\in\mathcal{E}_2} \Lambda \left(\frac{1}{\delta} \left(F_1(h_1,\ell_1) - F_1(h_2,\ell_2) \right) \right) \\
&= \delta \int_{-\infty}^{\infty} \text{sinc}^2(\delta\alpha) \, |S_1(\alpha)|^2 d\alpha \\
&\geq \frac{4}{\pi^2} \delta \int_{-1/2\delta}^{1/2\delta} |S_1(\alpha)|^2 d\alpha.
\end{aligned}$$

Lemma 7 *In the notation of Lemma 6, for any $K > 0$,*

$$\mathcal{N}(F_1, cF_2, \delta) \ll_K \mathcal{N}(F_1, F_1, \delta)^{1/2} \mathcal{N} \left(F_2, F_2, \frac{\delta}{|c|} \right)^{1/2}$$

if $K^{-1} \leq |c| \leq K$.

Proof. By Lemma 6 and the Cauchy-Schwarz inequality,

$$\begin{aligned}\mathcal{N}(F_1, cF_2, \delta) &\ll \delta \int_{-1/2\delta}^{1/2\delta} |S_1(\alpha)S_2(c\,\alpha)|d\alpha \\ &\ll_K \left(\delta \int_{-1/2\delta}^{1/2\delta} |S_1(\alpha)|^2 d\alpha\right)^{1/2} \left(\delta \int_{-|c|/2\delta}^{|c|/2\delta} |S_2(\alpha)|^2 d\alpha\right)^{1/2} \\ &\ll_K \mathcal{N}(F_1, F_1, \delta)^{1/2} \mathcal{N}\left(F_2, F_2, \frac{|\delta|}{c}\right)^{1/2}.\end{aligned}$$

Lemma 8 *Let $\tau \geq 1$ and suppose that the number of solutions $\mathcal{N}_{\mathcal{D}}(X, \Delta)$ of* (1.3) *satisfies*

$$\mathcal{N}_{\mathcal{D}}(X, \Delta) \ll_\tau 1 + X^\tau + \Delta X^2 \tag{2.10}$$

whenever $\mathcal{D}$ is a compact set satisfying (H1)–(H4). Then the number of solutions of

$$Q(\boldsymbol{m}) \asymp_{\mathcal{D}} X,\ 0 \leq Q(\boldsymbol{n}) - Q(\boldsymbol{m}) < \Delta X \tag{2.11}$$

is

$$\ll_{\mathcal{D},\tau} 1 + X^\tau + \Delta X^2$$

whenever $X > 0$, $\Delta > 0$ and $\mathcal{D}$ is a compact set satisfying (H1)–(H3).

In the following proof, implied constants may have the same dependencies as those in the conclusion of the lemma. We use this convention in subsequent proofs also.

Proof. Let $\mathcal{D}$ be given satisfying (H1)–(H3). Choose $f_1, \ldots, f_J$ as in Lemma 2 and fix j. As a consequence of Lemma 5, there is a compact set $\mathcal{D}^*$ satisfying (H1)–(H4) such that $\mathcal{C}_j$ is a arc of $\mathcal{C}^* = \partial\mathcal{D}^*$. Let L_1, L_2 be line segments joining $\mathbf{0}$ to the endpoints of $\mathcal{C}_j$ and let $\mathcal{E}_j$ be the part of $\mathcal{D}$ bounded by L_1, L_2, $\mathcal{C}_j$. Since $\mathcal{E}_j \subset \mathcal{D}^*$, the number of solutions of (1.3) with

$$\begin{aligned}&\boldsymbol{m} \in X^{1/2}2^{p/2}\mathcal{E}_j,\ \boldsymbol{n} \in X^{1/2}2^{p/2}\mathcal{E}_j,\\ &X2^{p-1} < Q(\boldsymbol{m}) \leq X2^p,\\ &X2^{p-1} < Q(\boldsymbol{n}) \leq X2^p\end{aligned}$$

is

$$\ll 1 + X^\tau + \Delta X^2 \ll 1 + X^\tau + \Delta X^2$$

for any integer $p \ll 1$. By Lemma 7, the number of solutions $\mathcal{N}_{j,k,p,q}$ of (1.3) with

$$\boldsymbol{m} \in X^{1/2}2^{p/2}\mathcal{E}_j,\ X2^{p-1} < Q(\boldsymbol{m}) \leq X2^p,$$
$$\boldsymbol{n} \in X^{1/2}2^{q/2}\mathcal{E}_k,\ X2^{q-1} < Q(\boldsymbol{n}) \leq X2^q$$

is also

$$\ll 1 + X^\tau + \Delta X^2.$$

when $p \ll 1$, $q \ll 1$.

We may evidently suppose that $\Delta < 1$. The number of solutions of (2.11) is

$$\leq \sum_p \sum_q \sum_{j=1}^J \sum_{k=1}^J \mathcal{N}_{j,k,p,q}$$

with summation extending over a bounded set of (p,q). Lemma 8 follows at once.

Lemma 9 *Let $K > 0$. Let $\tau \geq 1$ and suppose that the number of solutions of* (1.3) *is*

$$\ll_{\mathcal{D},\tau} 1 + X^\tau + \Delta X^2$$

whenever $X > 0$, $\Delta > 0$, and $\mathcal{D}$ is a compact set satisfying (H1)–(H4). Let $\mathcal{D}$ be given satisfying (H1)–(H3) and let $f_1, \dots, f_J$ be as in Lemma 2. Let $H \geq 1$. Then the number of solutions of

$$|G(f_j;\ell_1,h_1) + cG(f_k,\ell_2,h_2)| < \Delta H$$

with

$$(\ell_1,h_1) \in E(f_j) \cap \mathbb{Z}^2,\ h_1 \asymp_{f_j} H$$
$$(\ell_2,h_2) \in E(f_k) \cap \mathbb{Z}^2,\ h_2 \asymp_{f_k} H$$

is

$$\ll_{\mathcal{D},\tau,K} H^{2\tau} + \Delta H^4.$$

provided that $K^{-1} \leq |c| \leq K$.

Proof. By Lemma 7, we need only prove this in the case $j = k$, $|c| = 1$. To avoid trivialities we need only bound the number of solutions of

$$||G(f_j;\ell_1,h_1)| - |G(f_j;\ell_2,h_2)|| < \Delta H, h_1 \asymp H,\ h_2 \asymp H. \tag{2.12}$$

By Lemma 7 again, we can restrict (ℓ_1, h_1) and (ℓ_2, h_2) in (2.12) to a section S in such a way that the curve $\mathcal{C}_S$ given by

$$|G(f_j;u,v)| = 1, \quad (u,v) \in S \tag{2.13}$$

can be written $\{(x,h(x)) : c \le x \le d\}$ or $\{(h(x),x) : c \le x \le d\}$. Now Lemma 5 (in conjunction with Lemmas 3, 4) provides a compact set $\mathcal{D}^*$ satisfying (H1)–(H4) such that $\mathcal{C}_S$ is an arc of $\partial\mathcal{D}^*$.

For $(u,v) \in S$, we have

$$|G(f;u,v)| = Q(\mathcal{D}^*;(u,v))^{1/2}.$$

To see this, define $\tau > 0$ by $\frac{1}{\tau}(u,v) \in \mathcal{C}_S$, so that

$$Q(\mathcal{D}^*;(u,v)) = \tau^2, \quad \left|G\left(f;\frac{u}{\tau},\frac{v}{\tau}\right)\right| = 1.$$

Then

$$|G(f;u,v)| = \tau\left|G\left(f;\frac{u}{\tau},\frac{v}{\tau}\right)\right| = \tau = Q(\mathcal{D}^*;(u,v))^{1/2}.$$

Now (2.12) implies

$$Q(\mathcal{D}^*;\ell_1,h_1)) \asymp H^2,$$
$$Q(\mathcal{D}^*;(\ell_2,h_2)) - Q(\mathcal{D}^*;(\ell_1,h_1)) \ll H(\Delta H) = \Delta H^2.$$

There are

$$\ll 1 + (H^2)^\tau + \Delta(H^2)^2 \ll H^{2\tau} + \Delta H^4$$

such quadruples ℓ_1, h_1, ℓ_2, h_2.

Lemma 10 *Let $L \ge 1$. There are trigonometric polynomials*

$$P(y) = \sum_{0<|h|\le L} a_h e(hy)\ , \quad a_h \ll |h|^{-1}$$

and

$$Q(y) = \sum_{|h| \le L} b_h e(hy) \ , \quad b_h \ll L^{-1}$$

such that

$$|\psi(y) - P(y)| \le Q(y) \tag{2.14}$$

and

$$|\psi^*(y) - P(y)| \le Q(y). \tag{2.15}$$

Proof. See Vaaler [18] (also the appendix to Graham and Kolesnik [5]) for (2.14); the inequality (2.15) follows by a limit argument.

Lemma 10 reduces the study of $S(f, M)$, $S^*(f, M)$ to that of $S_h(f, M)$. We quote the result of applying the B-process to $S_h(f, M)$ from Kühleitner and Nowak [11].

Lemma 11 *Suppose that $h > 0$, $f^{(4)} : I \to \mathbb{R}$, $f^{(4)}$ is continuous on I and $f^{(2)}$ is never 0. In the notation introduced above,*

$$S_h(g, M) = \frac{M^{1/2}}{h^{1/2}} \sideset{}{''}\sum_{-hg'(b) \le m \le -hg'(a)} \frac{1}{\sqrt{g''\left(\phi\left(-\frac{m}{h}\right)\right)}} e(MG(f; m, h)) + O(r_h(a) + r_h(b) + \log 2M) \tag{2.16}$$

for $M \ge 1$. Here $\sum''$ indicates that values $m = -hg'(b)$, $m = -hg'(a)$ correspond to terms with weight $\frac{1}{2}$;

$$r_h(c) = \begin{cases} 0 \quad if \ \ hg'(c) \in \mathbb{Z} \\ \min\left(\frac{1}{\|hg'(c)\|}, \frac{M^{1/2}}{h^{1/2}}\right) \ \ otherwise. \end{cases}$$

In particular, the error term in (2.16) is $O(\log 2M)$ when $f'(a)$ and $f'(b)$ are rational.

The theory of the Riemann-Stieltjes integral used in the following lemma (and subsequently) is the version expounded in Apostol [1, Chapter 9]. In particular, the integral

$$I_1 = \int_v^w g(u) dh(u)$$

exists if $h : [v, w] \to \mathbb{C}$ is the sum of a continuous function and a step function continuous from the right, while $g : [v, w] \to \mathbb{C}$ is a function of bounded variation continuous from the left. When I_1 exists for given bounded functions g, h on $[v, w]$, so too does

$$I_2 = \int_v^w h(u) dg(u),$$

and

$$I_1 + I_2 = h(u)g(u)\Big|_v^w.$$

Lemma 12 *Let $\mathcal{D}$ be as in Theorem 3. Let $X \geq 1$. Let g, k be left-continuous functions of bounded variation on $[X, 2X]$. Then*

(i) $\displaystyle\int_X^{2X} g(w) dP_{\mathcal{D}}(w) \ll \|g\|_\infty X.$

(ii) If $|g(w)| \leq k(w)$, then

$$\int_X^{2X} g(w) dP_{\mathcal{D}}(w) \ll \int_X^{2X} k(w) dw + \left| \int_X^{2X} k(w) dP_{\mathcal{D}}(w) \right|.$$

(iii) If g is continuously differentiable, then

$$\int_X^{2X} g(w) dP_{\mathcal{D}}(w) \ll \|g\|_\infty X^{1/3} + \left| \int_X^{2X} g'(w) P_{\mathcal{D}}(w) dw \right|.$$

The sup norm is taken over $[X, 2X]$.

Proof. (i) This follows at once from

$$\int_X^{2X} g(w) dP_{\mathcal{D}}(w) = -m(\mathcal{D}) \int_X^{2X} g(w) dw + \int_X^{2X} g(w) dN_{\mathcal{D}}(w), \tag{2.17}$$

since

$$\left| \int_X^{2X} g(w) dN_{\mathcal{D}}(w) \right| \leq \|g\|_\infty (\mathcal{N}_{\mathcal{D}}(2X) - \mathcal{N}_{\mathcal{D}}(X)).$$

(ii) From (2.17),

$$\left|\int_X^{2X} g(w)dP_{\mathcal{D}}(w)\right| \le m(\mathcal{D})\int_X^{2X} k(w)dw + \int_X^{2X} k(w)dN_{\mathcal{D}}(w)$$
$$= 2m(\mathcal{D})\int_X^{2X} k(w)dw + \int_X^{2X} k(w)dP_{\mathcal{D}}(w).$$

(iii) The estimate $\|P_{\mathcal{D}}\|_\infty \ll X^{1/3}$ is due to van der Corput [4]. (Note that this implies $\|P^*_{\mathcal{D}}\|_\infty \ll X^{1/3}$.) We shall not need later refinements (the most recent is in Huxley [9]).

Now

$$\int_X^{2X} g(w)dP_{\mathcal{D}}(w) = g(w)P_{\mathcal{D}}(w)\Big|_X^{2X} - \int_X^{2X} g'(w)P_{\mathcal{D}}(w)dw$$
$$\ll \|g\|_\infty X^{1/3} + \left|\int_X^{2X} g'(w)P_{\mathcal{D}}(w)dw\right|.$$

The following lemma can be found in Titchmarsh [17] and Graham and Kolesnik [5].

Lemma 13 *Let F be a real differentiable function on $[a,b]$ and $G(w)$ a real continuous function on $[a,b]$. Suppose that $F'(w)/G(w)$ is monotonic and*

$$F'(w)/G(w) \ge m > 0$$

or

$$F'(w)/G(w) \le -m < 0$$

on $[a,b]$. Then

$$\left|\int_a^b G(w)e(F(w))dw\right| \le \frac{4}{m}.$$

We now state a proposition that can be used twice to obtain Theorem 3.

Proposition *Suppose that*

$$\mathcal{N}_{\mathcal{D}}(X,\Delta) \ll_{\mathcal{D},\tau} 1 + X^\tau + \Delta X^2$$

for some $\tau \ge 6/5+\epsilon$ and all compact sets $\mathcal{D}$ with the properties (H1)–(H3). Then for $0<\eta\le 1/11$, $\tau-\eta \ge 6/5+\epsilon$, we have

$$\mathcal{N}_{\mathcal{D}}(X,\Delta) \ll_{\mathcal{D},\tau,\epsilon} 1 + X^{\tau-\eta} + \Delta X^2 \tag{2.18}$$

for all compact sets $\mathcal{D}$ with the properties (H1)–(H3).

To deduce Theorem 3, we observe that the hypothesis of the proposition holds for $\tau = 4/3$, since for fixed $\boldsymbol{m}$, the number of solutions of (1.3) is

$$\begin{aligned} &\le \mathcal{N}_{\mathcal{D}}(Q(\boldsymbol{m}) + \Delta X) - \mathcal{N}_{\mathcal{D}}(Q(\boldsymbol{m}) - \Delta X) \\ &\ll_{\mathcal{D}} X^{1/3} + \Delta X \end{aligned}$$

by the result of [4]. We apply the proposition to show that the hypothesis of the proposition holds for $\tau = 4/3 - 1/11$. Applying the proposition again with $\eta = 4/3 - 1/11 - (6/5 + \epsilon)$, we obtain

$$\mathcal{N}_{\mathcal{D}}(X, \Delta) \ll_{\mathcal{D},\epsilon} 1 + X^{6/5+\epsilon} + \Delta X^2$$

whenever $\mathcal{D}$ is a compact set with the properties (H1)–(H3).

Proof of the Proposition. In view of Lemma 8, we need only prove (2.18) for a compact set $\mathcal{D}$ with the properties (H1)–(H4). Write $Q(\boldsymbol{m}) = Q(\mathcal{D}; \boldsymbol{m})$. The number of solutions of (1.3) can be written in the form

$$\begin{aligned} &\sum_{X < Q(\boldsymbol{m}) \le 2X} \{\mathcal{N}^*_{\mathcal{D}}(Q(\boldsymbol{m}) + \Delta X) - N^*_{\mathcal{D}}(Q(\boldsymbol{m}))\} \\ &= \int_X^{2X} \{N^*_{\mathcal{D}}(\omega + \Delta X) - N^*_{\mathcal{D}}(\omega)\} dN_{\mathcal{D}}(\omega) \\ &= \int_X^{2X} m(\mathcal{D}) \Delta X dN_{\mathcal{D}}(\omega) + \int_X^{2X} (P^*_{\mathcal{D}}(\omega + \Delta X) - P^*_{\mathcal{D}}(\omega)) dN_{\mathcal{D}}(\omega). \end{aligned}$$

Thus is suffices to show that

$$\int_X^{2X} (P^*_{\mathcal{D}}(\omega + \Delta X) - P^*_{\mathcal{D}}(\omega)) d\omega \ll \Delta X^2 \tag{2.19}$$

and

$$\int_X^{2X} P^*_{\mathcal{D}}(\omega_1) dP_{\mathcal{D}}(\omega) \ll X^{\tau - \eta} + \Delta X^2. \tag{2.20}$$

where $\omega_1 = \omega + \gamma$, $\gamma \in \{0, \Delta X\}$.

The bound (2.19) gives no trouble. We have

$$\int_X^{2X} (P_{\mathcal{D}}^*(\omega + \Delta X) - P_{\mathcal{D}}^*(\omega))d\omega$$
$$= \int_{X+\Delta X}^{2X+\Delta X} P_{\mathcal{D}}^*(\omega)d\omega - \int_X^{2X} P_{\mathcal{D}}^*(\omega)d\omega$$
$$= \left(\int_{2X}^{2X+\Delta X} - \int_X^{X+\Delta X}\right) P_{\mathcal{D}}^*(\omega)d\omega \ll X^{1/3}\Delta X.$$

Turning to (2.20), we rewrite the result of Lemma 2 as

$$P_{\mathcal{D}}^*(\omega) = \sum_{j=1}^{J} e_j S^*(f_j, \sqrt{\omega}) + F(\omega),$$

where $F(\omega)$ is a left-continuous function of bounded variation on $[X, 2X]$, and

$$F(\omega) = O(1).$$

Since

$$\int_X^{2X} F(\omega)dP_{\mathcal{D}}(\omega) \ll X$$

from Lemma 12 (i), we need only prove that

$$\int_X^{2X} S^*(f, \sqrt{\omega_1})dP_{\mathcal{D}}(\omega) \ll X^{\tau-\eta} + X^2\Delta \tag{2.21}$$

whenever $f = f_{\mathcal{D}} : [a, b] \to \mathbb{R}$ with $f^{(4)}$ continuous, $f^{(2)}$ is nowhere 0, and $f'(a)$, $f'(b)$ are rational.

We apply Lemma 10 with

$$L = X^{3/2-\tau+\eta}.$$

Writing

$$g_1(\omega) = S^*(f, \sqrt{\omega_1}),\ g_2(\omega) = \sum_{0<|h|\le L} a_h S_h(f, \sqrt{\omega_1})$$

and

$$k(\omega) = \sum_{|h|\le L} b_h S_h(f, \sqrt{\omega_1}),$$

so that

$$|g_1(\omega) - g_2(\omega)| \le k(\omega),$$

Lemma 7 (ii) gives

$$\int_X^{2X} g_1(\omega)dP_{\mathcal{D}}(\omega) - \int_X^{2X} g_2(\omega)dP_{\mathcal{D}}(\omega) \ll \int_X^{2X} k(\omega)d\omega + \left|\int_X^{2X} k(\omega)dP_{\mathcal{D}}(\omega)\right|. \tag{2.22}$$

The contribution to the right-hand side of (2.22) from $b_0 S_0(f, \sqrt{\omega_1})$ is

$$\ll X^{3/2}L^{-1} \ll X^{\tau-\eta}$$

from Lemma 12 (i).

We now apply Lemma 11. We see that it suffices to show that

$$\int_X^{2X} E(\omega)d\Gamma(\omega) \ll X^{\tau-\eta} + \Delta X^2,$$

where $d\Gamma(\omega)$ denotes either of $d\omega$, $dP_{\mathcal{D}}(\omega)$ and

$$E(\omega) = \omega_1^{1/4} \sum_{0<h\le L} h^{-3/2} \sum_{-hg'(a)\le \ell \le -hg'(b)} \kappa(h,\ell)e(\pm\omega_1^{1/2}G(\ell,h)) + O((\log X)^2).$$

Here $G(\ell,h) = G(f;\ell,h)$ and $|\kappa(h,\ell)| \ll 1$.

The integrals arising from the $O((\log X)^2)$ term in the expression for $E(\omega)$ are

$$O(X(\log X)^2)$$

by Lemma 12 (i), which is satisfactory. By a splitting-up argument, we need only show that, for either choice of $d\Gamma(\omega)$,

$$H^{-3/2} \sum_{(h,\ell)\in\mathcal{E}} \left|\int_X^{2X} \omega_1^{1/4} e(\omega_1^{1/2}G(\ell,h))d\Gamma(\omega)\right| \ll X^{\tau-\eta-\epsilon} + X^{2-\epsilon}\Delta. \tag{2.23}$$

Here we may suppose that ϵ is sufficiently small, and we have

$$\frac{1}{2} \le H \le X^{3/2-\tau+\eta}; \tag{2.24}$$

we define $\mathcal{E} = \mathcal{E}(f, H)$ by

$$\mathcal{E} = \{(h, \ell) \in \mathbb{Z}^2 : h \sim H,\ -hg'(a) \le \ell \le -hg'(b)\}. \tag{2.25}$$

The case $d\Gamma(\omega) = d\omega$ of (2.23) is immediate from Lemma 13. The left-hand side of (2.23) is

$$\ll H^{1/2} X^{1/4} (X^{-1/2} H)^{-1} \ll X^{3/4}.$$

For $d\Gamma(\omega) = dP_{\mathcal{D}}(\omega)$, we appeal to Lemma 12 (iii):

$$\begin{aligned} &\int_X^{2X} \omega_1^{1/4} e(\omega_1^{1/2} G(\ell, h)) dP_{\mathcal{D}}(\omega) \\ &\qquad \ll X^{1/4+1/3} + \left| \int_X^{2X} \omega_1^{-3/4} e(\omega_1^{1/2} G(\ell, h)) P_{\mathcal{D}}(\omega) d\omega \right| \\ &\qquad + \left| \int_X^{2X} \omega_1^{-1/4} G(\ell, h) e(\omega_1^{1/2} G(\ell, h)) P_{\mathcal{D}}(\omega) d\omega \right|. \end{aligned} \tag{2.26}$$

The second term on the right-hand side of (2.26) is also $O(X^{1/4+1/3})$, so that together with the first term the corresponding contribution to the left-hand side of (2.23) is

$$\begin{aligned} O(H^{1/2} X^{7/12}) &= O(X^{3/4+7/12-\tau/2+\eta/2}) \\ &= O(X^{\tau-\eta-\epsilon}) \end{aligned}$$

since $\tau - \eta > \frac{6}{5} > \frac{2}{3}\left(\frac{3}{4} + \frac{7}{12}\right)$. Thus we must show that

$$H^{-1/2} \sum_{(h,\ell)\in\mathcal{E}} \left| \int_X^{2X} \omega_1^{-1/4} e(\omega_1^{1/2} G(\ell, h)) P_{\mathcal{D}}(\omega) d\omega \right| \ll X^{\tau-\eta-\epsilon} + X^{2-\epsilon}\Delta.$$

We apply Lemma 2 again, noting that

$$\begin{aligned} H^{-1/2} \sum_{(h,\ell)\in\mathcal{E}} &\left| \int_X^{2X} \omega_1^{-1/4} e(\omega_1^{1/2} G(\ell, h)) O(1) d\omega \right| \\ &\ll H^{3/2} X^{3/4} \ll X^{3-3\tau/2+3\eta/2} \\ &\ll X^{\tau-\eta-\epsilon}, \end{aligned}$$

since

$$\frac{5\tau}{2} - \frac{5\eta}{2} \geq 3 + \epsilon.$$

Thus we must show that, with $f_1 : [a_1, b_1] \to \mathbb{R}$ a function (depending on $\mathcal{D}$) having the properties ascribed to f,

$$(2.27) \qquad H^{-1/2} \sum_{(h,\ell)\in\mathcal{E}} \left| \int_X^{2X} \omega_1^{-1/4} e(\omega_1^{1/2} G(h,\ell)) S(f_1, \sqrt{\omega}) d\omega \right|$$
$$\ll X^{\tau-\eta-\epsilon} + X^{2-\epsilon}\Delta.$$

We apply Lemma 10 again, with

$$L_1 = H^{3/2} X^{5/4-\tau+\eta+\epsilon}$$

in place of L. We obtain

$$(2.28) \qquad \left| \int_X^{2X} \omega_1^{-1/4} e(\omega_1^{1/2} G(\ell,h)) S(f_1, \sqrt{\omega}) d\omega \right|$$
$$\leq \left| \int_X^{2X} \omega_1^{-1/4} e(\omega_1^{1/2} G(\ell,h)) \sum_{0<|h'|\leq L_1} a_{h'} S_{h'}(f_1, \sqrt{\omega}) d\omega \right|$$
$$+ \int_X^{2X} \omega_1^{-1/4} \sum_{|h'|\leq L_1} b_{h'} S_{h'}(f_1, \sqrt{\omega}) d\omega.$$

The contribution to the right-hand side of (2.28) from b_0 is $O(X^{5/4} L_1^{-1})$. In bounding the left-hand side of (2.27), this gives rise to a contribution

$$\ll H^{3/2} X^{5/4} L_1^{-1} = X^{\tau-\eta-\epsilon}.$$

After a further splitting-up argument, it suffices to show that

$$(2.29) \qquad H^{-1/2} \sum_{(h,\ell)\in\mathcal{E}} \left| \int_X^{2X} \omega_1^{-1/4} e(\beta\omega_1^{1/2} G(\ell,h)) \sum_{h'\asymp K} c_{h'} S_{h'}(f_1, \sqrt{\omega}) d\omega \right|$$
$$\ll X^{\tau-\eta-2\epsilon} + X^{2-2\epsilon}\Delta$$

whenever $K \in \left[\frac{1}{2}, L\right]$, $c_{h'} \ll K^{-1}$ and $\beta \in \{0, 1\}$.

We apply Lemma 11 once more. The error $O(\log 2M)$ yields a contribution to the left-hand side of (2.29) that is

$$\ll H^{3/2} \int_X^{2X} \omega_1^{-1/4} \sum_{h' \asymp K} K^{-1} \log x d\omega$$
$$\ll H^{3/2} X^{3/4} \log X \ll X^{\tau-\eta-2\epsilon}$$

since

$$\frac{5\tau}{2} - \frac{5\eta}{2} \geq 3 + \frac{5\epsilon}{2}.$$

It remains to show that

(2.30)

$$H^{-1/2}K^{-3/2} \sum_{(h,\ell)\in\mathcal{E}} \sum_{(h',\ell')\in\mathcal{E}'} \left| \int_X^{2X} \omega_1^{-1/4}\omega^{1/4} e(\beta\omega_1^{1/2}G(\ell,h) + \omega^{1/2}G_1(\ell',h'))d\omega \right|$$
$$\ll X^{\tau-\eta-2\epsilon} + X^{2-2\epsilon}\Delta$$

for $\beta \in \{0, 1, -1\}$. Here, with $g_1 = \pm f_1$ having $g_1'' < 0$,

$$\mathcal{E}' = \{(h', \ell') \in \mathbb{Z}^2 : h' \sim K, -h'g_1'(a_1) \leq \ell' \leq -h'g'(b_1)\},$$

and $G_1(u, v) = G(f_1; u, v)$, while

$$\frac{1}{2} \leq K \leq L_1.$$

Now in (2.30),

$$\frac{d}{d\omega}\left(\beta\omega_1^{1/2}G(\ell,h) + \omega^{1/2}G_1(\ell',h')\right) \gg X^{-1/2}K$$

unless

(2.31) $$\beta = \pm 1, \quad H \asymp K.$$

If (2.31) does *not* hold, the left-hand side of (2.30) is

$$\ll H^{3/2}K^{1/2}(X^{-1/2}K)^{-1} \ll H^{3/2}X^{1/2},$$

which we have already seen is acceptable.

Suppose now that $\beta = \pm 1$ and $H \asymp K$. The contribution to the left-hand side of (2.30) from quadruples with

$$G(\ell, h) - G_1(\ell', h') \ll \Delta H \tag{2.32}$$

is estimated via Lemma 9, using a trivial bound for the integral, as

$$\begin{aligned} & H^{-1/2} K^{-3/2} X (H^{2\tau} + H^4 \Delta) \\ & \ll H^{2\tau - 2} X + H^2 X \Delta \\ & \ll X^{(3/2 - \tau + \eta)(2\tau - 2) + 1} + X^{4 - 2\tau + 2\eta} \Delta. \end{aligned} \tag{2.33}$$

Now

$$4 - 2\tau + 2\eta \leq 4 - 12/5 < 2 - 2\epsilon.$$

Moreover

$$\begin{aligned} \left(\frac{3}{2} - \tau + \eta\right)(2\tau - 2) + 1 &= 5\tau - 2\tau^2 + \eta(2\tau - 2) - 2 \\ &\leq \tau - \eta - 2\epsilon, \end{aligned}$$

since $\tau > 6/5 + \eta$,

$$\begin{aligned} 4\tau - 2\tau^2 - 2 &< 4\left(\frac{6}{5} + \eta\right) - 2\left(\frac{6}{5} + \eta\right)^2 - 2 \\ &< -\frac{4\eta}{5} - \frac{2}{25} \leq -\frac{5\eta}{3} - 3\epsilon \leq -\eta(2\tau - 1) - 2\epsilon. \end{aligned} \tag{2.34}$$

This shows that the bound in (2.33) is satisfactory.

If $\alpha = \pm 1$, $H \asymp K$ and (2.32) does *not* hold, say $|G(\ell, h) - G_1(\ell', h')| > C\Delta H$, $C = C(\mathcal{D}) > 0$, then

$$\begin{aligned} & \left| \frac{d}{d\omega} \left(G(\ell, h)\omega_1^{1/2} - G_1(h', \ell')\omega^{1/2} \right) \right| \\ & \quad = \left| \frac{1}{2} \left(G(h, \ell) - G_1(h', \ell') \right) \omega^{-1/2} \right| + O(\Delta H X^{-1/2}) \\ & \quad \gg |G(h, \ell) - G_1(h', \ell')| X^{-1/2}. \end{aligned}$$

Consider the contribution to the left-hand side of (2.30) from quadruples with

$$\delta H < |G(h,\ell) - G_1(h',\ell')| \le 2\delta H,$$

where $\delta = C\Delta 2^{k-1}$, $k = 1, 2, \ldots$, $\delta \ll 1$. This contribution is

$$\ll H^{-2}(H^{2\tau} + H^4\delta)\min(X, (\delta H X^{-1/2})^{-1})$$

$$\ll H^{2\tau-2}X + HX^{1/2}.$$

Summing over $O(\log X)$ values of Δ, the quadruples for which (2.32) fails contribute

$$\ll H^{2\tau-2}X\log X + HX^{1/2}\log X.$$

The second term was shown earlier to be satisfactory, and the calculation leading to (2.34) gives the same outcome for the first term. This completes the proof of the proposition.

Proof of Theorem 4. By a splitting-up argument and Minkowski's inequality, it suffices to show that

$$\int_T^{2T} \left| \sum_{Q(\boldsymbol{m})\sim X} Q(\boldsymbol{m})^{-\sigma-it} \right|^2 dt \ll T^{1+\epsilon/2}. \tag{2.35}$$

The left-hand side of (2.35) is

$$\sum_{\substack{\boldsymbol{m} \\ Q(\boldsymbol{m})\sim X}} \sum_{\substack{\boldsymbol{n} \\ Q(\boldsymbol{n})\sim X}} (Q(\boldsymbol{n})Q(\boldsymbol{m}))^{-\sigma} \int_T^{2T} (Q(\boldsymbol{m})/Q(\boldsymbol{n}))^{it} dt \tag{2.36}$$

$$\le 4X^{-2\sigma} \sum_{\substack{\boldsymbol{m} \\ Q(\boldsymbol{m})\sim X}} \sum_{\substack{\boldsymbol{n} \\ Q(\boldsymbol{m})\le Q(\boldsymbol{n})\le X}} \min\left(T, \frac{1}{\log\frac{Q(\boldsymbol{n})}{Q(\boldsymbol{m})}}\right).$$

Those $\boldsymbol{m}$, $\boldsymbol{n}$ with

$$\left(\log\frac{Q(\boldsymbol{n})}{Q(\boldsymbol{m})}\right)^{-1} < 4$$

contribute

$$\ll X^{-2\sigma}\left\{\sum_{\substack{\boldsymbol{n} \\ Q(\boldsymbol{n})\le 2X}} 1\right\}^2 \ll X^{2-2\sigma} \ll T$$

to the left-hand side of (2.36).

For $4 \le U \le T$, let

$$M(U) = |\{(\boldsymbol{m}, \boldsymbol{n}) : X < Q(\boldsymbol{m}) \le Q(\boldsymbol{n}) \le 2X,$$
$$(\log Q(\boldsymbol{n})/Q(\boldsymbol{m}))^{-1} \ge U\}|,$$

where $|\dots|$ denotes cardinality and we agree that $(\log 1)^{-1} \ge U$. If $(\boldsymbol{m}, \boldsymbol{n})$ is counted in $M(U)$,

$$\frac{U}{2} \le \frac{Q(\boldsymbol{m})}{Q(\boldsymbol{n}) - Q(\boldsymbol{m})} \le \frac{2X}{Q(\boldsymbol{n}) - Q(\boldsymbol{m})},$$
$$0 \le Q(\boldsymbol{n}) - Q(\boldsymbol{m}) \le \frac{4X}{U}.$$

By Theorem 3,

$$M(U) \ll X^{6/5+\epsilon/3} + U^{-1}X^2.$$

A splitting-up argument now yields the following bound for the left-hand side of (2.36):

$$\ll T + \sum_{U = 2^{-\ell}T \ge 4} X^{-2\sigma} U M(U)$$
$$\ll T + (X^{6/5+\epsilon/3-2\sigma}T + X^{2-2\sigma}) \log T$$

since $\ell = 0, 1, 2, \dots$ runs over $O(\log T)$ values. The last upper bound is

$$\ll T^{1+\epsilon},$$

since $X^{2-2\sigma} \le T$ and $X^{6/5+\epsilon/3-2\sigma} \le X^{\epsilon/3} \ll T^{2\epsilon/3}$. This completes the proof of Theorem 4.

§3 Proof of Theorem 2.

The following simple result is Lemma 5 of [3].

Lemma 14 *Let $A > 0$, $A < B \le 2A$, $C \ge 2$, $C < D \le 2C$. Let f be a bounded measurable function on $[A, B]$. Then*

$$\int_C^D \left| \int_A^B f(x) x^{it} dx \right|^2 dt \ll A \log C \int_A^B |f(x)|^2 dx.$$

Lemma 15 *Let $F : [c, d] \to \mathbb{R}$. Suppose F is continuously differentiable and $|F'(u)| \geq k|u|$ $(c \leq u \leq d)$ for some $k > 0$. Then for $\gamma > 0$,*

$$E = \{u \in [c, d] : |F(u)| \leq \gamma\}$$

is the union of at most 2 intervals of length $O_k(\gamma^{1/2})$.

Proof. Suppose first that $c \geq 0$. After changing the sign of F if necessary,

$$F'(u) \geq ku > 0 \quad \text{on} \quad (c, d].$$

Clearly $\{u \in [c, d] : -\gamma \leq F(u) \leq \gamma\}$ is empty or is a single interval with endpoints C, D say. Moreover,

$$\begin{aligned} 2\gamma \geq F(D) - F(C) &= \int_C^D F'(u)du \geq k \int_C^D u\, du \\ &= \frac{k}{2}(D^2 - C^2) \geq \frac{k}{2}(D - C)^2, \end{aligned}$$

$$D - C \leq 2\gamma^{1/2}k^{-1/2}.$$

If $c < 0$, we treat the interval $[c, \min(d, 0)]$ in the same way by replacing $F(u)$ by $F(-u)$. This completes the proof.

Lemma 16 *Let $Y > 1$, $L \geq Y^3$. Let $f : [a, b] \to \mathbb{R}$ and suppose that f'' is continuous and never 0. Let*

$$S_L(\omega) = \frac{1}{2\pi i} \sum_{n\omega^{-1/2} \in [a,b]} \sum_{0 < |h| \leq L} h^{-1} e\left(h\omega^{1/2} f\left(\frac{n}{\omega^{1/2}}\right)\right).$$

Then

$$\int_Y^{2Y} |S(f, \sqrt{\omega}) - S_L(\omega)|^2 d\omega \ll_f Y. \tag{3.1}$$

Proof. The left-hand side of (3.1) is at most

$$2\int_Y^{2Y} \left| \psi(\omega^{1/2} f(0)) - \sum_{0 < |h| \leq L} \frac{e(h\omega^{1/2} f(0))}{2\pi i h} \right|^2 d\omega \tag{3.2}$$

$$+ 2\int_Y^{2Y} \left| \sum_{\substack{n \in [aY^{1/2}, b(2Y)^{1/2}] \\ n > 0}} G_n(\omega) g_n(\omega) \right|^2 d\omega$$

(where the first summand may be omitted if $a > 0$). Here

$$G_n(\omega) = \sum_{|h|>L} \frac{e\left(h\omega^{1/2} f\left(\frac{n}{\omega^{1/2}}\right)\right)}{2\pi i h},$$

$$g_n(\omega) = \begin{cases} 1 & \text{if } n\omega^{-1/2} \in [a,b] \\ 0 & \text{otherwise.} \end{cases}$$

The first summand in (3.2) is $O(Y)$ by the bounded convergence of the Fourier series of ψ. Applying Minkowski's inequality, it suffices to show that, writing $I(n) = \{\omega \in [Y, 2Y] : \frac{n}{\omega^{1/2}} \in [a,b]\}$,

$$\sum_{n \in (aY^{1/2}, b(2Y)^{1/2}]} \int_{I(n)} |G_n(\omega)|^2 d\omega \ll Y^{1/2}. \tag{3.3}$$

We begin the proof of (3.3) by noting that

$$\int_{Z_1}^{Z_2} \left| \sum_{|h|>L} \frac{e(hz)}{h} \right|^2 dz \ll \frac{Z_2 - Z_1 + 1}{L} \quad (Z_2 > Z_1), \tag{3.4}$$

by Parseval's equality on any interval of length 1. We now fix $n \in (aY^{1/2}, b(2Y)^{1/2}]$ and write $I = I(n)$. Let I_1 be the set of ω in I for which

$$\left| \frac{d}{d\omega} \left(\omega^{1/2} f\left(\frac{n}{\omega^{1/2}}\right)\right) \right| \geq \beta\omega^{-1/2}; \tag{3.5}$$

the positive number β will be chosen below. We shall see that I_1 is the union of at most three intervals, on each of which the derivative in (3.5) has constant sign. It follows from (3.4) that

$$\int_{I_1} |G_n(\omega)|^2 \left| \frac{d}{d\omega} \left(\omega^{1/2} f\left(\frac{n}{\omega^{1/2}}\right)\right) \right| d\omega \ll \frac{Y^{1/2}}{L}$$

so that

$$\int_{I_1} |G_n(\omega)|^2 d\omega \ll \frac{Y^{1/2}}{L(\beta Y^{-1/2})} = \frac{Y}{L\beta}.$$

Now

$$\frac{d}{d\omega} \left(\omega^{1/2} f\left(\frac{n}{\omega^{1/2}}\right)\right) = \frac{1}{2}\,\omega^{-1/2} \left\{ f\left(\frac{n}{\omega^{1/2}}\right) - \frac{n}{\omega^{1/2}}\, f'\left(\frac{n}{\omega^{1/2}}\right) \right\}.$$

We observe that

$$\left|\frac{d}{du}\{f(u)-uf'(u)\}\right| = |-uf''(u)| \geq |u| \text{ on } [a,b],$$

where $k = k(f) > 0$. By Lemma 15, $I\backslash I_1$ may be written

$$I\backslash I_1 = \left\{\omega \in I : \frac{n}{\omega^{1/2}} \in E\right\},$$

where E is the union of at most two intervals of length $O(\beta^{1/2})$ with endpoints between $\frac{n}{(2Y)^{1/2}}$ and $\frac{n}{Y^{1/2}}$. It may readily be verified that $I\backslash I_1$ is the union of at most two intervals of length $O\left(\frac{Y^{3/2}\beta^{1/2}}{|n|}\right)$. Again using bounded convergence,

$$\int_{I\backslash I_1} |G_n(\omega)|^2 d\omega \ll \frac{Y^{3/2}\beta^{1/2}}{|n|}.$$

Choosing $\beta = Y^{-1/3}L^{-2/3}|n|^{2/3}$, we see that

$$\int_I |G_n(\omega)|^2 d\omega \ll \frac{Y^{4/3}}{L^{1/3}|n|^{2/3}}.$$

Thus the left-hand side of (3.3) is

$$\ll \sum_{1\leq |n|\leq b(2Y)^{1/2}} \frac{Y^{4/3}}{L^{1/3}|n|^{2/3}} \ll \frac{Y^{3/2}}{L^{1/3}} \ll Y^{1/2}.$$

This completes the proof of the lemma.

Proof of Theorem 2. Let $\sigma > 1$. We have, for $X > 0$,

$$(3.6) \quad Z_{\mathcal{D}}(s) = \sum_{Q(\boldsymbol{m})\leq X} Q(\boldsymbol{m})^{-s} + \int_X^\infty \frac{dN_{\mathcal{D}}(\omega)}{\omega^s}$$

$$= \sum_{Q(\boldsymbol{m})\leq X} Q(\boldsymbol{m})^{-s} + m(\mathcal{D})\int_X^\infty \omega^{-s} d\omega + \int_X^\infty \frac{dP_{\mathcal{D}}(\omega)}{\omega^s}$$

$$= \sum_{Q(\boldsymbol{m})\leq X} Q(\boldsymbol{m})^{-s} + \frac{m(\mathcal{D})X^{1-s}}{s-1} - \frac{P_{\mathcal{D}}(X)}{X^s} + s\int_X^\infty \frac{P_{\mathcal{D}}(\omega)d\omega}{\omega^{s+1}}.$$

Using $P_{\mathcal{D}}(\omega) \ll \omega^{1/3}$, this formula provides the analytic continuation of $Z_{\mathcal{D}}(s)$ to the half-plane $\sigma > 1/3$; we note the simple pole at 1 with residue $m(\mathcal{D})$.

Let $T \geq 2$. In proving Theorem 2, we may suppose that $3/5 \leq \sigma \leq 3/4$. Define X by

$$X^{\sigma+1/4} = T.$$

Note that $X^{2-2\sigma} \leq X^{\sigma+1/4} = T$. From the last expression in (3.6),

$$\int_T^{2T} |Z_{\mathcal{D}}(\sigma+it)|^2 dt \ll T + \int_T^{2T} \left| \sum_{Q(\boldsymbol{m}) \leq X} Q(\boldsymbol{m})^{-\sigma-it} \right|^2 dt$$
$$+ T^2 \int_T^{2T} \left| \int_X^\infty \frac{P_{\mathcal{D}}(\omega)}{\omega^{\sigma+it+1}} \, d\omega \right|^2 dt.$$

In view of Theorem 4, we need only show that

$$\int_T^{2T} \left| \int_X^\infty \frac{P_{\mathcal{D}}(\omega)}{\omega^{\sigma+it}} \, d\omega \right|^2 dt \ll T^{-1+\epsilon}. \tag{3.7}$$

Let

$$F_j(t) = \int_{X2^{j-1}}^{X2^j} \frac{P_{\mathcal{D}}(\omega)}{\omega^{\sigma+it+1}} \, d\omega \quad (j \geq 1).$$

It suffices to show that

$$\int_T^{2T} |F_j(t)|^2 dt \ll T^{-1+\epsilon} j^{-4} \quad (j \geq 1). \tag{3.8}$$

For then (3.7) follows from Cauchy's inequality:

$$\int_T^{2T} \left| \sum_{j=1}^\infty F_j(t) \right|^2 dt \leq \int_T^{2T} \left(\sum_{j=1}^\infty j^{-2} \right) \left(\sum_{j=1}^\infty j^2 |F_j(t)|^2 \right) dt$$
$$\leq \sum_j \int_T^{2T} j^2 |F_j(t)|^2 dt \ll T^{-1+\epsilon}.$$

Arguments of the following kind will be used implicitly. Suppose that

$$F_j = \sum_{k=1}^K F_{j,k} \ , \quad K = O((\log T)^C)$$

for an absolute constant C. Then

$$\int_T^{2T} |F_j(t)|^2 dt \le K \int_T^{2T} \sum_{k=1}^{K} |F_{j,k}(t)|^2 dt.$$

Thus to prove (3.8) it suffices to show that

$$\int_T^{2T} |F_{j,k}(t)|^2 dt \ll T^{-1+\epsilon/2} j^{-4} \quad (1 \le k \le K).$$

(It is harmless to split F_j into $O((\log T)^C)$ parts.)

We begin by noting that if $G(\omega)$ is a bounded measurable function on

$$J(j) = [X2^{j-1}, X2^j],$$

then

$$\int_T^{2T} \left| \int_{J(j)} \frac{G(\omega)}{\omega^{\sigma+it+1}}\, d\omega \right|^2 dt \ll \log T (X2^j)^{-2\sigma} \|G\|_\infty^2 \tag{3.9}$$

from Lemma 14. In particular, if $G(\omega) = O(T^{\epsilon/6})$, the last quantity is

$$\ll T^{-1+\epsilon/2} j^{-4}.$$

Recalling Lemma 2, we need only show that

$$\int_T^{2T} \left| \int_{J(j)} \frac{S(f, \sqrt{\omega})}{\omega^{\sigma+it+1}}\, d\omega \right|^2 dt \ll T^{-1+\epsilon} j^{-4} \tag{3.10}$$

where $f_{\mathcal{D}} = f : I = [a, b] \to \mathbb{R}$, $f^{(4)}$ is continuous, $f^{(2)}$ is never 0, and $f'(a)$, $f'(b)$ are rational.

In the notation of Lemma 16, with $L = (X2^j)^{1/3}$, we have

$$\begin{aligned} &\int_T^{2T} \left| \int_{J(j)} \frac{S(f, \sqrt{\omega})}{\omega^{\sigma+it+1}}\, d\omega \right|^2 dt \\ &\le 2 \int_T^{2T} \left| \int_{J(j)} \frac{S(f, \sqrt{\omega}) - S_L(\omega)}{\omega^{\sigma+it+1}}\, d\omega \right|^2 dt \\ &\qquad + 2 \int_T^{2T} \left| \int_{J(j)} \frac{S_L(\omega)}{\omega^{\sigma+it+1}}\, d\omega \right|^2 dt. \end{aligned}$$

The first summand on the right is

$$\ll (X2^j)^{-2\sigma} \log T \ll T^{-1+\epsilon} 2^{-2j\sigma},$$

by Lemma 14 in tandem with Lemma 16. It remains to show that

$$\int_T^{2T} \left| \int_{J(j)} \frac{S_L(\omega)}{\omega^{\sigma+it+1}} \, d\omega \right|^2 dt \ll T^{-1+\epsilon} j^{-4}.$$

We apply a splitting-up argument to the variable h in $S_L(\omega)$, followed by Lemma 11. The integral corresponding to the term

$$\sum_{|h|\sim H} O(|h|^{-1} \log T) = O((\log T)^2)$$

that arises from (2.16) can be bounded satisfactorily via the estimate (3.9). Thus it remains to show that

$$H^{-3} \int_T^{2T} \left| \int_{J(j)} \omega^{-\sigma-\alpha i t-3/4} \sum_{(h,\ell)\in\mathcal{E}} b(h,\ell) e(G(\ell,h)\omega^{1/2}) d\omega \right|^2 dt \ll T^{-1+\epsilon/2} j^{-4} \tag{3.11}$$

with $\mathcal{E}$ as in (2.25), $|b(h,\ell)| \le 1$, and $\alpha \in \{-1, 1\}$. Here $1 \le H \le (X2^j)^3$.

Now

$$\begin{aligned} &\left| \frac{d}{d\omega} \left(-\frac{\alpha t \log \omega}{2\pi} + G(\ell,h)\omega^{1/2} \right) \right| \\ &\qquad = \left| -\frac{\alpha t}{2\pi\omega} + \frac{1}{2} G(\ell,h)\omega^{-1/2} \right| \gg H(X2^j)^{-1/2} \end{aligned}$$

unless $\alpha = e^*$ (the sign of G) and

$$T(X2^j)^{-1} \asymp H(X2^j)^{-1/2},$$

that is,

$$H \asymp T(X2^j)^{-1/2}. \tag{3.12}$$

Suppose for a moment that either $\alpha \neq e^*$ or that (3.12) does not hold. Lemma 13 yields

$$\int_{J(j)} \omega^{-\sigma-\alpha it-3/4} e(G(\ell,h)\omega^{1/2}) d\omega$$
$$\ll (X2^j)^{-1/4-\sigma} H^{-1}.$$

Since $|\mathcal{E}|^2 \ll H^4$, the left-hand side of (3.11) is

$$\ll H^{-1}T(X2^j)^{-1/2-2\sigma}$$
$$\ll T^{-1}j^{-4}$$

from the definition of X.

Now suppose that $\alpha = e^*$ and that (3.12) holds. The left-hand side of (3.11) is estimated via Lemma 14 as

(3.13)

$$\ll H^{-3}X2^j \log T \int_{J(j)} \omega^{-2\sigma-3/2} \left| \sum_{(h,\ell)\in\mathcal{E}} e(G(\ell,h)\omega^{1/2}) \right|^2 d\omega$$

$$\ll H^{-3}X2^j \log T \sum_{(h_1,\ell_1)\in\mathcal{E}} \sum_{(h_2,\ell_2)\in\mathcal{E}} \int_{J(j)} \omega^{-2\sigma-3/2} e(\omega^{1/2}(G(h_1,\ell_1) - G(h_2,\ell_2))) d\omega.$$

Consider first the contribution to the right-hand side of (3.13) from quadruples h_1, ℓ_1, h_2, ℓ_2 with

$$|G(\ell_1,h_1) - G(\ell_2,h_2)| < H^{-3/5}.$$

There are $O(H^{12/5+\epsilon/4})$ such quadruples, by Theorem 3 and Lemma 9. Estimating the integral trivially, these quadruples contribute

$$\ll H^{-3/5+\epsilon/4}X2^j \log T(X2^j)^{-2\sigma-1/2}$$
$$\ll (X2^j)^{-2\sigma+1/2}H^{-3/5}T^{\epsilon/2}$$
$$\ll (X2^j)^{-2\sigma+4/5}T^{-3/5+\epsilon/2} \ll T^{-1+\epsilon/2}j^{-4}$$

since

$$(X2^j)^{2\sigma-4/5} \geq (X2^j)^{2/5(\sigma+1/4)} \gg T^{2/5}j^4.$$

Now consider the contribution to the right-hand side of (3.11) from quadruples with

$$\delta H \le |G(\ell_1, h_1) - G(\ell_2, h_2)| < 2\delta H,$$

where $\delta = H^{-8/5} 2^{k-1}$, $k = 1, 2, \ldots$, $\delta \ll 1$. There are $O(\delta H^4)$ such quadruples, again by Theorem 3 and Lemma 9. Estimating the integral via Lemma 13, these quadruples contribute

$$\begin{aligned} &\ll \delta H\, X 2^j \log T (X2^j)^{-2\sigma - 3/2} (\delta H)^{-1} (X 2^j)^{1/2} \\ &\ll (X2^j)^{-2\sigma} \log T \ll T^{-1} j^{-4}. \end{aligned}$$

Thus quadruples with

$$|G(\ell_1, h_1) - G(\ell_2, h_2)| \ge H^{-8/5}$$

contribute $O(T^{-1+\epsilon/2} j^{-4})$ to the right-hand side of (3.13). This completes the proof of Theorem 2.

§4 Proof of Theorem 1.

For the convenience of the reader, we repeat some arguments from Huxley and Nowak [10] without much change. Let $y = y(x)$ be a large positive number, $y(x) < x^{1/2}$, to be chosen below. We have

$$\begin{aligned} (4.1)\quad \mathcal{A}_{\mathcal{D}}(x) &= \sum_{\substack{0 < Q(m_1, m_2) \le x \\ \gcd(m_1, m_2) = 1}} 1 \\ &= \sum_{0 < Q(m_1, m_2) \le x} \ \sum_{d \,|m_1, d|\, m_2} \mu(d) \\ &= \sum_{d \ge 1} \mu(d) N_{\mathcal{D}} \left(\frac{\sqrt{x}}{d} \right) \\ &= \sum_{d \le y} \mu(d) P_{\mathcal{D}} \left(\frac{\sqrt{x}}{d} \right) + \sum_{d > y} \mu(d) P_{\mathcal{D}} \left(\frac{\sqrt{x}}{d} \right) + m(\mathcal{D}) x \sum_{d=1}^{\infty} \frac{\mu(d)}{d^2} \\ &= E_1(x) + E_2(x) + \frac{6}{\pi^2}\, m(\mathcal{D}) x, \end{aligned}$$

say.

We now quote a formula of Perron type from [10]:

$$\sum_{d>y} \mu(d) N_{\mathcal{D}}\left(\frac{x}{d^2}\right) = \frac{1}{2\pi i}\int_{3-ix^5}^{3+ix^5} Z_{\mathcal{D}}(s) f_y(2s)\,\frac{x^s}{s}\,ds + O(x^{1/3+\epsilon}). \tag{4.2}$$

Here

$$f_y(s) = \frac{1}{\zeta(s)} - \sum_{m\le y}\frac{\mu(m)}{m^s}\,.$$

Since we assume R. H., we have

$$f_y(\sigma+it) = \sum_{n>y}\frac{\mu(n)}{n^{\sigma+it}} \ll y^{1/2-\sigma+\epsilon}(|t|^\epsilon + 1) \tag{4.3}$$

for $\sigma \ge 1/2+\epsilon$. This is obtained by a slight variant of the proof of [17, Theorem 14.25(A)].

By a slight adaptation of the application of the residue theorem in §4 of [10], we find that

$$\begin{aligned} &\frac{1}{2\pi i}\int_{3-ix^5}^{3+ix^8} Z_{\mathcal{D}}(s) f_y(2s)\,\frac{x^s}{s}\,ds \\ &\qquad = \frac{1}{2\pi i}\int_{\frac{3}{5}-ix^5}^{\frac{3}{5}+ix^5} Z_{\mathcal{D}}(s) f_y(2s)\,\frac{x^s}{s} + m(\mathcal{D}) x f_y(2) + O(1). \end{aligned} \tag{4.4}$$

We may combine (4.2) and (4.4) to obtain

$$\begin{aligned} E_2(x) &= \sum_{d>y}\mu(d) N_{\mathcal{D}}\left(\frac{x}{d^2}\right) - m(\mathcal{D}) x f_y(2) \\ &= \frac{1}{2\pi i}\int_{\frac{3}{5}-ix^5}^{\frac{3}{5}+ix^5} Z_{\mathcal{D}}(s) f_y(2s)\,\frac{x^s}{s}\,ds + O(x^{1/3+\epsilon}). \end{aligned}$$

After a splitting-up argument and an application of (4.3),

$$E_2(x) \ll y^{1/2-6/5} x^{3/5+\epsilon/2}\left(T^{-1}\int_T^{2T} |Z_{\mathcal{D}}(\sigma+it)|\,dt + 1\right) + x^{1/3+\epsilon}$$

for some T, $2 \le T \le x^5$. By Theorem 2 and Cauchy's inequality,

$$E_2(x) \ll y^{-7/10} x^{3/5+\epsilon} + x^{1/3+\epsilon}.$$

We choose y so that $y^{-7/10} x^{3/5} = x^{5/13}$, that is,

$$y = x^{4/13}.$$

It remains to show that

$$E_1(x) \ll x^{5/13+\epsilon}.$$

With the notation of Lemma 2, we have an expression of the form

$$E_1(x) = \sum_{d \le y} \mu(d) \sum_{j=1}^{J} e_j S\left(f_j, \frac{x^{1/2}}{d}\right) + O(y).$$

Accordingly, it suffices to show that, for $D \in [1/2, y]$,

$$\sum_{d \sim D} \mu(d) S\left(f, \frac{x^{1/2}}{d}\right) \ll x^{5/13+\epsilon/2} \tag{4.5}$$

whenever $f_{\mathcal{D}} = f : I = [a, b] \to \mathbb{R}$, $f^{(4)}$ is continuous, $f^{(2)}$ is never 0, and $f'(a)$, $f'(b)$ are rational.

We apply Lemma 10, with $L = x^{1/2-5/13} = x^{3/26}$:

$$\sum_{d \sim D} \mu(d) S\left(f, \frac{x^{1/2}}{d}\right) = \sum_{d \sim D} \sum_{0<|h| \le L} \mu(d) a_h S_h \left(f, \frac{x^{1/2}}{d}\right) + O\left(\sum_{d \sim D} \sum_{|h| \le L} b_h S_h \left(f, \frac{x^{1/2}}{d}\right)\right). \tag{4.6}$$

By the choice of L, the contribution to the right-hand side of (4.6) from b_0 is $O(x^{5/13})$. I now show that

$$\sum_{d \sim D} \sum_{0<|h| \le L} \mu(d) a_h S_h \left(f, \frac{x^{1/2}}{d}\right) \ll x^{5/13+\epsilon/2}. \tag{4.7}$$

It will be clear from the discussion that the proof of

$$\sum_{d \sim D} \sum_{0<|h| \le L} b_h S_h \left(f, \frac{x^{1/2}}{d}\right) \ll x^{5/13+\epsilon/2}$$

is simpler. So once we prove (4.7), the proof of Theorem 1 will be complete. By a splitting-up argument, it suffices to show that

$$\sum_{h\sim H} a_h \sum_{d\sim D} \mu(d) S_h\left(f, \frac{x^{1/2}}{d}\right) \ll x^{5/13+\epsilon/3} \tag{4.8}$$

whenever

$$a_h \ll H^{-1},\ \frac{1}{2} \le H \le x^{3/26},\ \frac{1}{2} \le D \le x^{4/13}. \tag{4.9}$$

We apply the B-process (Lemma 11). The contribution from the term $O(\log 2M)$ to the left-hand side of (4.8) is

$$\ll \sum_{h\sim H} h^{-1} \sum_{d\sim D} \log x \ll x^{4/13} \log x.$$

We have thus reduced the proof to showing that

$$H^{-3/2} \sum_{(h,\ell)\in\mathcal{E}} c(h,\ell) \sum_{d\sim D} \left(\frac{x}{d^2}\right)^{1/4} \mu(d) e\left(\frac{G(\ell,h)x^{1/2}}{d}\right) \ll x^{5/13+\epsilon/3}, \tag{4.10}$$

with $\mathcal{E}$ as in (2.25) and $|c(h,\ell)| \le 1$.

We first treat the case

$$H > D^{5/3} x^{-35/78} \tag{4.11}$$

of (4.10) by a method similar to that of Zhai [21]. Let

$$S = \sum_{(h,\ell)\in\mathcal{E}} c(h,\ell) \sum_{d\sim D} f_d\, e\left(\frac{G(\ell,h)x^{1/2}}{d}\right),$$

where $f_d = \frac{\mu(d)D^{1/2}}{d^{1/2}} \ll 1$. Let Q be a natural number, we partition $[-CH, CH]$ $(C > 0, C = C(\mathcal{D}))$ into subintervals $I_1, \ldots, I_Q$ of equal length. Thus

$$|S| \le \sum_{q=1}^{Q} \sum_{d\sim D} \left| \sum_{G(\ell,h)\in I_q} c(h,\ell) e\left(\frac{G(\ell,h)x^{1/2}}{d}\right) \right|.$$

(Summation conditions $(h,\ell) \in \mathcal{E}$ are implicit here and below.) Cauchy's inequality gives

$$(4.12) \quad |S|^2 \le QD \sum_{q=1}^{Q} \sum_{\substack{G(\ell,h)\in I_q \\ G(\ell',h')\in I_q}} \left| \sum_{d\sim D} e\left(\frac{(G(\ell,h)-G(\ell',h'))}{d} x^{1/2} \right) \right|$$

$$\le QD \sum_{|G(\ell,h)-G(\ell',h')|\le \frac{2CH}{Q}} \left| \sum_{d\sim D} e\left(\frac{(G(\ell,h)-G(\ell',h'))}{d} x^{1/2} \right) \right|.$$

For quadruples with

$$|G(\ell,h) - G(\ell',h')| < H^{-3/5},$$

we estimate

$$S(h,\ell,h',\ell') = \sum_{d\sim D} e\left(\frac{G(\ell,h)-G(\ell',h')}{d} x^{12} \right)$$

trivially. There are $O(H^{12/5+\epsilon})$ such quadruples by Theorem 3 and Lemma 9, giving

$$(4.13) \qquad \sum_{h,\ell,h',\ell'} S(h,\ell,h',\ell') \ll H^{12/5+\epsilon} D$$

for these quadruples.

Now consider quadruples with

$$(4.14) \qquad \delta H \le |G(\ell,h) - G(\ell',h')| < 2\delta H,$$

where $\delta = H^{-8/5}2^{k-1}$, $k = 0, 1, \ldots$, $\delta \le 2C/Q$. For these quadruples, the exponent pair $\left(\frac{1}{2}, \frac{1}{2}\right)$ gives the estimate

$$S(h,\ell,h',\ell') \ll \left(\frac{\delta H x^{1/2}}{D} \right)^{1/2} + (\delta H x^{1/2} D^{-2})^{-1}.$$

(See [5] for the theory of exponent pairs.) Again by Theorem 3 and Lemma 9, there are $O(\delta H^4)$ quadruples satisfying (4.14). For these quadruples,

$$(4.15) \qquad \sum_{h,\ell,h',\ell'} S(h,\ell,h',\ell') \ll \delta H^4 \left(\frac{\delta H x^{1/2}}{D} \right)^{1/2} + \delta H^4 (\delta H x^{1/2} D^{-2})^{-1}$$

$$\ll Q^{-3/2} H^{9/2} x^{1/4} D^{-1/2} + H^3 x^{-1/2} D^2.$$

Note that

$$H^3x^{-1/2}D^2 \ll H^{12/5}D,$$

since

$$HD \ll x^{3/26+4/13} < x^{1/2}.$$

Hence we can combine (4.12), (4.13) and (4.15) (summed over $O(\log x)$ values of δ) to obtain

$$|S|^2 \ll QD(H^{12/5+\epsilon}D + Q^{-3/2}H^{9/2}x^{1/4}D^{-1/2}\log x),$$

$$S \ll x^{\epsilon/3}(Q^{1/2}DH^{6/5} + Q^{-1/4}H^{9/4}x^{1/8}D^{1/4}).$$

Minimizing this expression over $Q \in [1, \infty)$ in the usual way, we obtain

$$\begin{aligned} S &\ll DH^{6/5}x^{\epsilon/3} + (DH^{6/5})^{1/3}(H^{9/4}x^{1/8}D^{1/4})^{2/3}x^{\epsilon/3} \\ &\ll x^{\epsilon/3}(DH^{6/5} + D^{1/2}H^{19/10}x^{1/12}). \end{aligned}$$

Accordingly, the left-hand side of (4.10) is

$$\ll H^{-3/2}D^{-1/2}x^{1/4+\epsilon/3}(DH^{6/5} + D^{1/2}H^{19/10}x^{1/12}).$$

By the lower bound (4.11) imposed on H,

$$H^{-3/10}D^{1/2}x^{1/4+\epsilon/3} \ll x^{5/13+\epsilon/3}.$$

Moreover,

$$H^{2/5}x^{1/3+\epsilon/3} \ll x^{148/390+\epsilon/3} \ll x^{5/13}$$

from (4.9). This gives the desired bound in the case (4.11).

For the smaller values of H we need a lemma.

Lemma 17 *Let $H \geq 1$, $N \geq 1/2$, $\Delta > 0$. The number of solutions of*

$$\left|\frac{G(\ell, h)}{n} - \frac{G(\ell', h')}{n'}\right| < \frac{\Delta H}{N} \tag{4.16}$$

with $(h, \ell) \in \mathcal{E}$, $(h', \ell') \in \mathcal{E}$, $n \sim N$, $n' \sim N$ is

$$O(H^{16/5+\epsilon}N\log 2N + \Delta H^2N^4).$$

Proof. Let $1 \le d \le N$. We estimate the number of solutions of (4.16) with $(n, n') = d$, say $\mathcal{N}(d)$. Fix such a pair n, n'. We apply Theorem 3 in conjunction with Lemma 9, with $c = -\frac{n}{n'}$. The number of quadruples h, ℓ, h', ℓ' satisfying (4.16) is

$$\ll H^{12/5+\epsilon} + \Delta H^4.$$

Summing over n,n',

$$\mathcal{N}(d) \ll \left(\frac{N}{d}\right)^2 (H^{12/5+\epsilon} + \Delta H^4). \tag{4.17}$$

On the other hand, we may fix h, ℓ, h', ℓ' and observe that (4.16) implies

$$\left|\frac{G(\ell, h)}{G(\ell', h')} - \frac{n}{n'}\right| \ll \Delta. \tag{4.18}$$

Since the n/n' are spaced apart at least $(N/d)^{-2}$, the number of solutions of (4.18) is

$$\ll \frac{\Delta}{(N/d)^2} + 1.$$

Summing over h, ℓ, h', ℓ',

$$\mathcal{N}(d) \ll H^4 + \Delta H^4 N^2 d^{-2}.$$

This can be combined with (4.17) to obtain

$$\mathcal{N}(d) \ll H^{16/5+\epsilon} N d^{-1} + \Delta H^4 N^2 d^{-2}.$$

Summing over d gives the bound claimed in the lemma.

Completion of the proof of Theorem 1. To prove (4.10) when

$$H \le D^{5/3} x^{-35/78}, \tag{4.19}$$

we use a standard decomposition of sums

$$\sum_{d \sim D} \mu(d) F(d)$$

(where F is any complex function on $[D, 2D]$); see [12] or [2]. The sum can be decomposed into $O(D^{\epsilon/6})$ sums of the forms

$$\sum_{\substack{m \sim M,\, n \sim N \\ mn \sim D}} a_m F(mn), \tag{I}$$

with

$$N \gg D^{2/3}\ ,\ |a_m| \le 1;$$

and

(II) $$\sum_{\substack{m\sim M,\, n\sim N\\ mn\sim D}} a_m b_n F(mn),$$

with

$$D^{1/3} \ll N \ll D^{1/2}\ ,\ |a_m| \le 1\ ,\ |b_n| \le 1.$$

In (4.10), we have

$$F(d) = d^{-1/2} \sum_{(h,\ell)\in\mathcal{E}} c(h,\ell) e\left(\frac{G(\ell,h)x^{1/2}}{d}\right),$$

so that sums of type I take the form

$$S_I = M^{-1/2} \sum_{(h,\ell)\in\mathcal{E}} c(h,\ell) \sum_{m\sim M} a_m \sum_{\substack{n\sim N\\ mn\sim D}} n^{-1/2} e\left(\frac{G(\ell,h)x^{1/2}}{mn}\right)$$

with $|a_m| \le 1$, and sums of type II take the form

$$S_{II} = D^{-1/2} \sum_{(h,\ell)\in\mathcal{E}} c(h,\ell) \sum_{m\sim M} a_m \sum_{\substack{n\sim N\\ mn\sim D}} b_n e\left(\frac{G(\ell,h)x^{1/2}}{mn}\right),$$

with $|a_m| \le 1$, $|b_n| \le 1$. We have to show that

(4.20) $$S_I \ll H^{3/2} x^{5/13-1/4+\epsilon/6} = H^{3/2} x^{7/52+\epsilon/6}$$

for $N \gg D^{2/3}$, and

(4.21) $$S_{II} \ll H^{3/2} x^{7/52+\epsilon/6}$$

for $D^{1/3} \ll N \ll D^{1/2}$. The case that determines the exponent in Theorem 1 will turn out to be (4.21) with $D = x^{4/13}$, $H = D^{5/3} x^{-35/78}$ and any N, $D^{1/3} \ll N \ll D^{1/2}$.

We begin with (4.20). By a partial summation argument it suffices to show that, for fixed $(h,\ell) \in \mathcal{E}$,

$$D^{-1/2} \sum_{m\sim M} a_m \sum_{\substack{N\le n<u\\ mn\sim D}} e\left(\frac{G(\ell,h)x^{1/2}}{mn}\right) \ll H^{-1/2}x^{7/52+\epsilon/6} \tag{4.22}$$

for $u \in [N, 2N]$. We apply Theorem 4 of [2] with $\alpha = \beta = -1$, $X \asymp \frac{Hx^{1/2}}{D}$, $\kappa = \lambda = 1/2$. The left-hand side of (4.22) is

$$\ll (\log x)^2 D^{-1/2}(DN^{-1/2} + DX^{-1} + (D^6X^2N^{-2})^{1/8}). \tag{4.23}$$

We observe that $X \gg D^{1/2} \gg N$, since $x^{1/2} > D^{3/2}$. Since $N \gg D^{2/3}$, the bound in (4.23) is

$$\ll (\log x)^2(D^{1/6} + H^{1/4}x^{1/8}D^{-1/6}).$$

Recalling (4.19),

$$\begin{aligned} D^{1/6}(H^{-1/2}x^{7/52})^{-1} &\ll Dx^{-35/156-7/52} \\ &\ll x^{4/13-35/156-7/52} \ll 1. \end{aligned}$$

Moreover,

$$H^{1/4}x^{1/8}D^{-1/6}(H^{-1/2}x^{7/52})^{-1} \ll D^{13/12}x^{-36/104} \ll 1.$$

Thus (4.23) is a satisfactory bound and we pass on to (4.21).

By a standard device (see for example pp. 49–50 of Harman [6]) it suffices to show that

$$\begin{aligned} S'_{II} = \sum_{(h,\ell)\in\mathcal{E}} c(h,\ell) \sum_{m\sim M} a_m \sum_{n\sim N} b_n e\left(\frac{G(\ell,h)x^{1/2}}{mn}\right) \\ \ll D^{1/2}H^{3/2}x^{7/52+\epsilon/7}. \end{aligned} \tag{4.24}$$

Let

$$R = \max\left(1, \frac{Hx^{1/2+\epsilon/8}}{NM^2}\right).$$

We partition $\left[-\frac{C'H}{N}, \frac{C'H}{N}\right]$ (where $C' = C'(\mathcal{D}) > 0$) into R subintervals $J_1, \ldots, J_R$ of equal length. We have

$$|S'_{II}| \le \sum_{m\sim M} \sum_{r=1}^{R} \left| \sum_{\frac{G(\ell,h)}{n} \in J_r} c(h,\ell) e\left(\frac{G(\ell,h)x^{1/2}}{nm}\right) \right|,$$

suppressing summation conditions $(h,\ell) \in \mathcal{E}$, $n \sim N$ here and below. By Cauchy's inequality,

$$\text{(4.25)} \qquad \begin{aligned} |S'_{II}|^2 &\le MR \sum_{r=1}^{R} \sum_{\frac{G(h,\ell)}{n}, \frac{G(h',\ell')}{n'} \in J_r} S(\boldsymbol{v}) \\ &\le MR \sum_{\left|\frac{G(h,\ell)}{n} - \frac{G(h',\ell')}{n'}\right| \le \frac{2C'H}{NR}} S(\boldsymbol{v}). \end{aligned}$$

Here $\boldsymbol{v} = (h, \ell, h', \ell', n, n')$,

$$S(\boldsymbol{v}) = \sum_{m\sim M} e\left(\left(\frac{G(\ell,h)}{n} - \frac{G(\ell',h')}{n'}\right)\frac{x^{1/2}}{m}\right).$$

By Lemma 17 there are $O(H^{16/5+\epsilon} N \log 4N)$ vectors $\boldsymbol{v}$ for which

$$\left|\frac{G(\ell,h)}{n} - \frac{G(\ell',h')}{n'}\right| < H^{9/5}N^{-2}.$$

Estimating $S(\boldsymbol{v})$ trivially, we find that

$$\text{(4.26)} \qquad \sum_{\boldsymbol{v}} S(\boldsymbol{v}) \ll x^{\epsilon/8} H^{16/5} NM$$

for these $\boldsymbol{v}$.

Now consider those $\boldsymbol{v}$ with

$$\text{(4.27)} \qquad \frac{\delta H}{N} \le \left|\frac{G(\ell,h)}{n} - \frac{G(\ell',h')}{n'}\right| < \frac{2\delta H}{N},$$

where

$$\delta = H^{9/5}N^{-2}2^{k-1},\ k = 1, 2, \ldots,\ \delta \le 2C'/R.$$

Again by Lemma 17, there are $O(\delta H^4 N^2 x^{\epsilon/8})$ of these $\boldsymbol{v}$. We can apply the Kusmin-Landau estimate [5, Theorem 2.1] to obtain

$$S(\boldsymbol{v}) \ll \left(\frac{\delta H}{N} \cdot \frac{x^{1/2}}{M^2}\right)^{-1},$$

since

$$\begin{aligned}\frac{d}{dm}\left(\left(\frac{G(\ell,h)}{n} - \frac{G(\ell',h')}{n'}\right)\frac{x^{1/2}}{m}\right) &\ll \frac{\delta H x^{1/2}}{NM^2} \\ &\ll \frac{Hx^{1/2}}{NM^2R} \ll x^{-\epsilon/8}.\end{aligned}$$

Thus those $\boldsymbol{v}$ with (4.27) satisfy

$$\begin{aligned}\sum_{\boldsymbol{v}} S(\boldsymbol{v}) &\ll \delta H^4 N^2 \left(\frac{\delta H x^{1/2}}{NM^2}\right)^{-1} x^{\epsilon/8} \\ &\ll H^3 x^{-1/2+\epsilon/8} N^3 M^2 \\ &\ll H^{16/5} NM\end{aligned} \tag{4.28}$$

since

$$N^2 M \ll D^{3/2} < x^{1/2-\epsilon/8}.$$

We conclude from (4.25), (4.26), (4.28) that

$$\begin{aligned}|S'_{II}|^2 &\ll RH^{16/5}NM^2x^{\epsilon/8} \\ &\ll x^{\epsilon/4}(H^{16/5}NM^2 + H^{21/5}x^{1/2}).\end{aligned}$$

Since $N \gg D^{1/3}$,

$$S'_{II} \ll x^{\epsilon/8}(H^{8/5}D^{5/6} + H^{21/10}x^{1/4}). \tag{4.29}$$

Now

$$\begin{aligned}&H^{8/5}D^{5/6}(D^{1/2}H^{3/2}x^{7/52})^{-1} \\ &\qquad = H^{1/10}D^{1/3}x^{-7/52} \\ &\qquad \le D^{1/2}x^{-7/156-7/52} < D^{1/2}x^{-2/13} < 1,\end{aligned}$$

while

$$H^{21/10}x^{1/4}(D^{1/2}H^{3/2}x^{7/52})^{-1}$$
$$= H^{3/5}D^{-1/2}x^{3/26} \le D^{1/2}x^{-2/13} \le 1.$$

Thus (4.29) is a satisfactory estimate for S'_{II}, and the proof of Theorem 1 is complete.

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