Outcome A: Recall and apply the formula for arc length of a space curve.

The **arc length** of the space curve parameterized by the differentiable vector function 
\( \vec{r}(t) = (f(t), g(t), h(t)), a \leq t \leq b, \) is

\[
L = \int_a^b \| \vec{r}'(t) \| \, dt
= \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2} \, dt
= \int_a^b \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 + \left( \frac{dz}{dt} \right)^2} \, dt.
\]

If we think of \( \vec{r}'(t) \) as the velocity of a particle moving along the curve \( \vec{r}(t), \) then \( \| \vec{r}'(t) \| \) is the speed of the particle, and the arc length formula says that the distance traveled is the integral of the speed times time.

**Example.** The arc length the space curve parameterized by \( \vec{r}(t) = (12t, 8t^{3/2}, 3t^2), 0 \leq t \leq 1, \) is

\[
L = \int_0^1 \sqrt{[12]^2 + [12t^{1/2}]^2 + [6t]^2} \, dt = \int_0^1 \sqrt{144 + 144t + 36t^2} \, dt
= 6 \int_0^1 \sqrt{4 + 4t + t^2} \, dt = 6 \int_0^1 \sqrt{(t + 2)^2} \, dt
= 6 \int_0^1 (t + 2) \, dt \quad \text{[no absolute value because } t + 2 \geq 0 \text{ on } 0 \leq t \leq 1]\]
\[
= 6 \left[ \frac{t^2}{2} + 2t \right]_0^1 = 6 \left[ \frac{1}{2} + 2 \right] = 15.
\]

Here is a picture of this space curve whose arc length is 15.
Outcome B: Find the arc length function and reparameterize a vector function by arc length.

Example. The curve parameterized by \( \vec{r}_1(t) = (t, t^2, t^3), 1 \leq t \leq 2 \) is the same as the curve parameterized by \( \vec{r}_2(u) = (e^u, e^{2u}, e^{3u}), 0 \leq u \leq \ln 2 \).

This is because the one-to-one change of variable \( t = e^u \) gives \( t^2 = e^{2u}, \ t^3 = e^{3u} \), with \( t = 1 \) corresponding to \( u = 0 \) and \( t = 2 \) corresponding to \( u = \ln 2 \).

The one-to-one change of variable in this Example is called a reparameterization of the curve.

We can reparameterize a curve by its arc length.

Suppose a curve \( C \) is the graph of \( \vec{r}(t) = (f(t), g(t), h(t)), a \leq t \leq b \), where \( \vec{r}' \) is continuous, and \( C \) is traversed exactly once as \( t \) increases from \( t = a \) to \( t = b \).

The arc length function \( s \) is defined by
\[
s(t) = \int_a^t \| \vec{r}'(u) \| \, du = \int_a^t \sqrt{\left[f'(u)\right]^2 + \left[g'(u)\right]^2 + \left[h'(u)\right]^2} \, du.
\]

By the Fundamental Theorem of Calculus, we have
\[
\frac{ds}{dt} = \| \vec{r}'(t) \|.
\]

The assumption of traversing the curve exactly once implies that \( \| \vec{r}'(t) \| > 0 \) except at finitely many values of \( t \) in \([a, b]\).

This means that the arc length \( s \) is an invertible function of \( t \): there is a function \( t = t(s) \) that is the inverse of \( s = s(t) \).

The function \( t = t(s) \) says that one unit of arc length along the curve is achieved at the value \( t = t(1) \), two units of arc length are achieved at \( t = t(2) \), etc.

The composition \( \vec{r}(t(s)) \) is the reparameterization of the curve by arc length.

NOTE: it is generally impossible to find explicitly the inverse function needed for this reparameterization by arc length.

Example. The arc length function for \( \vec{r}(t) = (12t, 8t^{3/2}, 3t^2), 0 \leq t \leq 1 \), is
\[
s(t) = 6 \int_0^t (u + 2) \, du = 6 \left[ \frac{u^2}{2} + 2u \right]_0^t = 6 \left[ \frac{t^2}{2} + 2t \right] = 3t^2 + 12t.
\]

We find the inverse of this arc length function by the quadratic formula:
\[
t = \frac{-12 + \sqrt{144 + 12s}}{6}, \quad 0 \leq s \leq 15.
\]

The reparameterization of the curve by arc length is
\[
\vec{r}(t(s)) = \left\langle 12\left(\frac{-12 + \sqrt{144 + 12s}}{6}\right), 8\left(\frac{-12 + \sqrt{144 + 12s}}{6}\right)^{3/2}, 3\left(\frac{-12 + \sqrt{144 + 12s}}{6}\right)^2 \right\rangle.
\]
Outcome C: Recall and apply the definition of curvature for a smooth curve.

A parameterization $\vec{r}(t)$ of a curve is called smooth on an interval $I$ if $\vec{r}'$ is continuous and $\vec{r}'(t) \neq 0$ on $I$.

A curve is called smooth if it has a smooth parameterization.

NOTE: smooth curves have continuous tangent vectors, i.e., no corners or cusps.

Recall that the unit tangent vector is $\vec{T}(t) = \vec{r}'(t)/\|\vec{r}'(t)\|$, which is continuous and never zero for a smooth curve.

The notion of curvature is a measurement of how quickly the curve changes direction, i.e., the magnitude of the rate of change of the unit tangent vector with respect to arc length.

The curvature of a smooth curve is the quantity

$$\kappa = \left\| \frac{d\vec{T}}{ds} \right\| = \left\| \frac{d\vec{T}}{ds/dt} \right\| / \|\vec{r}'(t)\|^3.$$ 

Example. For the circle $\vec{r}(t) = \langle a \cos t, a \sin t, 0 \rangle$ of center $(0,0,0)$ and radius $a$ lying in the $xy$-plane, we have

$$\vec{r}'(t) = \langle -a \sin t, a \cos t, 0 \rangle,$$

$$\vec{r}''(t) = \langle -a \cos t, -a \sin t, 0 \rangle,$$

Thus $\|\vec{T}'(t)\| = 1$, and so $\kappa(t) = 1/a$, the reciprocal of the radius.

Outcome D: Compute the curvature of a smooth curve using the formula on p.880.

The geometric definition of curvature is awkward computationally. Here is a much more computationally friendly formula.

Theorem. The curvature of a smooth space curve is

$$\kappa(t) = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3}.$$ 

The proof of this is in the appendix of this lecture note.

Example. For $\vec{r}(t) = \langle 12t, 8t^{3/2}, 3t^2 \rangle$, we have

$$\vec{r}'(t) = \langle 12, 12t^{1/2}, 6t \rangle,$$

$$\vec{r}''(t) = \langle 0, 6t^{-1/2}, 6 \rangle,$$

and so

$$\vec{r}'(t) \times \vec{r}''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 12 & 12t^{1/2} & 6t \\ 0 & 6t^{-1/2} & 6 \end{vmatrix} = \langle 36t^{1/2}, -72, 72t^{-1/2} \rangle.$$ 

The curvature is

$$\kappa(t) = \frac{\sqrt{1296t + 5184 + 5184t^{-1}}}{(144 + 144t + 36t^2)^{3/2}}.$$
At the point on the curve corresponding to \( t = 1/2 \), the curvature is

\[
\kappa(1/2) = \frac{\sqrt{16200}}{(225)^{3/2}} \approx 0.0377,
\]
i.e., the curve is bending at \( t = 1/2 \) as though it were on a circle of radius \( 1/\kappa(1/2) \approx 26.5 \).

Outcome E: Find the principal unit normal vector, binormal vector, the normal plane, and the osculating plane for a smooth curve.

There are two more unit vectors, in addition to the unit tangent vector, associated to a smooth curve.

The **principal unit vector**, or simply the **unit normal** is

\[
\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|},
\]
which is orthogonal to \( \vec{T}(t) \) because \( 1 = \|\vec{T}(t)\|^2 = \vec{T}(t) \cdot \vec{T}(t) \) differentiates to \( 0 = 2\vec{T}(t) \cdot \vec{T}'(t) \).

The **binormal vector** is

\[
\vec{B}(t) = \vec{T}(t) \times \vec{N}(t),
\]
which is orthogonal to both vectors \( \vec{T}(t) \) and \( \vec{N}(t) \) of length one, and because the angle between \( \vec{T}(t) \) and \( \vec{N}(t) \) is \( \pi/2 \), has a length of 1.

The **normal plane** of a smooth curve \( \vec{r}(t) \) at a point \( \vec{r}(t_0) \) is the plane determined by the normal and binormal vectors, i.e., by the unit tangent vector \( \vec{T}(t_0) \) (as a vector normal to the plane) and the point \( \vec{r}(t_0) \).

The **osculating** plane of a curve \( \vec{r}(t) \) at a point \( \vec{r}(t_0) \) is the plane determined by the unit tangent vector and the principle unit vector, i.e, by the binormal vector \( \vec{B}(t_0) \) (as the vector normal to the plane) and the point \( \vec{r}(t_0) \).

Example. Here is a picture with the three unit vectors associated to \( \vec{r}(t) = \langle 12t, 8t^{3/2}, 3t^2 \rangle \), \( 0 \leq t \leq 1 \), at five different points on the curve.

![Diagram of unit vectors and osculating plane](image-url)
Appendix. The proof of the formula for curvature is a matter of showing that

\[ ||\vec{T}'(t)|| = \frac{||\vec{r}'(t) \times \vec{r}''(t)||}{||\vec{r}'(t)||^2}. \]

Well, \( \vec{T} = \vec{r}'/||\vec{r}'|| \) and \( ||\vec{r}'|| = ds/dt \), so that

\[ \vec{r}' = ||\vec{r}'||\vec{T} = \frac{ds}{dt}\vec{T}. \]

Differentiation of this (by the Product Rule) gives

\[ \vec{r}'' = \frac{d^2s}{dt^2}\vec{T} + \frac{ds}{dt}\vec{T}'. \]

The cross product of the first and second derivatives of \( \vec{r} \) is then

\[ \vec{r}' \times \vec{r}'' = \frac{ds}{dt}\vec{T} \times \left( \frac{d^2s}{dt^2}\vec{T} + \frac{ds}{dt}\vec{T}' \right) = \frac{ds}{dt} \frac{ds}{dt} \vec{T} \times \vec{T} + \left( \frac{ds}{dt} \right)^2 \vec{T} \times \vec{T}'. \]

Since the angle between \( \vec{T} \) and \( \vec{T}' \) is 0, the formula \( ||\vec{a} \times \vec{b}|| = ||\vec{a}|| \ ||\vec{b}|| \sin \theta \) implies that \( \vec{T} \times \vec{T}' = \vec{0} \). Differentiation of 1 = \( ||\vec{T}(t)||^2 = \vec{T}(t) \cdot \vec{T}(t) \) gives 0 = \( 2\vec{T}(t) \cdot \vec{T}'(t) \). Thus the angle between \( \vec{T} \) and \( \vec{T}' \) is \( \theta = \pi/2 \), and so the formula \( ||\vec{a} \times \vec{b}|| = ||\vec{a}|| \ ||\vec{b}|| \sin \theta \) implies that

\[ ||\vec{r}' \times \vec{r}''|| = \left( \frac{ds}{dt} \right)^2 ||\vec{T}|| \ ||\vec{T}'|| = \left( \frac{ds}{dt} \right)^2 ||\vec{T}'||. \]

Dividing both sides by \( (ds/dt)^2 \) and substituting \( ||\vec{r}'(t)|| \) for \( ds/dt \) gives the result. \( \square \)