Outcome A: Recall and apply the definition of limit of a function of several variables.

Let $f$ be a function of two variables whose domain $D$ contains points arbitrarily close to the point $(a, b)$.

We say the limit of $f(x, y)$ as $(x, y)$ approaches $(a, b)$ (within the domain $D$) is the number $L$ and we write

$$L = \lim_{(x,y)\to(a,b)} f(x, y),$$

if for every number $\epsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all points $(x, y) \in D$ within a distance of $\delta$ from $(a, b)$ there holds

$$|f(x, y) - L| < \epsilon.$$

There are infinitely many ways for a point $(x, y)$ to approach $(a, b)$: straight line approaches, quadratic and cubic approaches, squiggly approaches, spiral approaches, etc. These are illustrated for $(x, y)$ approaching $(0, 0)$.

When the limit $L$ exists for $f(x, y)$ as $(x, y)$ approaches $(a, b)$, EVERY approach of $(x, y)$ towards $(a, b)$ gives the same limiting value for $f$.

On the other hand, when there are two different approaches of $(x, y)$ towards $(a, b)$ that give different limiting values of $f$, then the limit of $f$ as $(x, y)$ approaches $(a, b)$ does not exist.

Examples. Find the limit, if it exists, or show that the limit does not exist.

(a) $\lim_{(x,y)\to(1,0)} \ln \left( \frac{1 + y^2}{x^2 + xy} \right)$.

Since the natural log function is continuous on its domain, we have

$$\lim_{(x,y)\to(1,0)} \ln \left( \frac{1 + y^2}{x^2 + xy} \right) = \ln \left( \lim_{(x,y)\to(1,0)} \frac{1 + y^2}{x^2 + xy} \right),$$

provided the limit on the right exists and is bigger than 0.
The numerator and denominator of the fraction are continuous functions, and so by the quotient rule for limits we have
\[
\lim_{(x,y)\to(1,0)} \frac{1 + y^2}{x^2 + xy} = \frac{\lim_{(x,y)\to(1,0)} 1 + y^2}{\lim_{(x,y)\to(1,0)} x^2 + xy} = \frac{1}{1} = 1.
\]
Thus we obtain the limit
\[
\lim_{(x,y)\to(1,0)} \ln \left( \frac{1 + y^2}{x^2 + xy} \right) = \ln(1) = 0.
\]
(b) \(\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^4 + y^6}\).
Both the numerator and the denominator evaluate to 0 as \((x, y)\) approaches \((0, 0)\), and so we have a 0/0 situation (but no two variable l’Hospital’s rule unfortunately).
If we suspect the limit does not exist, we choose different approaches to see if we get different limiting values.
Straight-line approaches are determined by \(y = mx\) for a constant \(m\), where \((x, y) = (x, mx)\) approaches \((0, 0)\) as \(x \to 0\).
This substitution of \(y = mx\) simplifies the limit:
\[
\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^4 + y^6} = \lim_{x \to 0} \frac{x(mx)^3}{x^4 + (mx)^6} = \lim_{x \to 0} \frac{m^3x^4}{x^4(1 + m^6x^2)} = \lim_{x \to 0} \frac{m^3}{1 + m^6x^2} = m^3.
\]
Thus different straight-line approaches give different limiting values, and so the limit does not exist.
(c) \(\lim_{(x,y)\to(0,0)} \frac{x^2ye^y}{x^4 + 4y^2}\).
Approaching \((0, 0)\) along the \(x\)-axis, i.e., setting \(y = 0\), gives
\[
\lim_{(x,0)\to(0,0)} \frac{x^2ye^y}{x^4 + 4y^2} = \lim_{(x,0)\to(0,0)} \frac{0}{x^4} = 0.
\]
Approaching \((0, 0)\) along the \(y\)-axis, i.e., setting \(x = 0\), gives
\[
\lim_{(0,y)\to(0,0)} \frac{x^2ye^y}{x^4 + 4y^2} = \lim_{(0,y)\to(0,0)} \frac{0}{4y^2} = 0.
\]
We might suspect that the limit could be 0, but along the quadratic approach \(y = x^2\), i.e., \((x, x^2) \to (0, 0)\) as \(x \to 0\), we get
\[
\lim_{(x,y)\to(0,0)} \frac{x^2ye^y}{x^4 + 4y^2} = \lim_{x \to 0} \frac{x^2x^2e^{x^2}}{x^4 + 4(x^2)^2} = \lim_{x \to 0} \frac{x^4e^{x^2}}{5x^4} = \lim_{x \to 0} \frac{e^{x^2}}{5} = \frac{1}{5} \neq 0.
\]
The limit does not exist because there are two different approaches that give different limiting values.
We probe the limit by the straight-line approaches \( y = mx \) which gives

\[
\lim_{(x,y) \to (0,0)} \frac{x^2 \sin^2 y}{x^2 + 2y^2} = \lim_{x \to 0} \frac{x^2 \sin^2 mx}{x^2 + 2mx^2} = \lim_{x \to 0} \frac{\sin^2 mx}{1 + 2m^2} = 0.
\]

We might suspect that the limit exists and is equal to 0.

To justify this, we notice that since \( 0 \leq \frac{x^2 \sin^2 y}{x^2 + 2y^2} \leq 1 \), we have the inequalities

\[
0 \leq \frac{x^2 \sin^2 y}{x^2 + 2y^2} \leq \sin^2 y.
\]

The limits of the outer two functions as \( (x, y) \to (0, 0) \) are both 0, and so the Squeeze Theorem tells us that

\[
\lim_{(x,y) \to (0,0)} \frac{x^2 \sin^2 y}{x^2 + 2y^2} = 0.
\]

The notion of the limit of a function of two variables readily extends to functions of three or more variables.

Outcome B: Recall and apply the definition of continuity of a function of several variables.

A function \( f \) of two variables is **continuous at a point** \( (a, b) \) in its domain \( D \) if

\[
\lim_{(x,y) \to (a,b)} f(x, y) = f(a, b).
\]

A function \( f \) is **continuous on its domain** \( D \) if it is continuous at every point \( (a, b) \) of its domain \( D \).

These definitions extend readily to functions of three or more variables.

Example. Every polynomial in \( n \) variables is continuous on its domain \( D = \mathbb{R}^n \).

Example. Find the set of points on which \( f(x, y, z) = \frac{\sqrt{y}}{x^2 - y^2 + z^2} \) is continuous.

For \( f \) to be continuous, the numerator gives us the restriction \( y \geq 0 \), and the denominator gives us the restriction \( x^2 - y^2 + z^2 \neq 0 \).

That is, the function is discontinuous when \( y < 0 \) or when \( x^2 - y^2 + z^2 = 0 \), i.e., the point \( (x, y, z) \) lies on the cone \( y^2 = x^2 + z^2 \).