Outcome A: Evaluate the line integral of a function along a piecewise smooth curve with respect to arc length.

Let $\vec{r}(t)$, $a \leq t \leq b$, be a smooth curve $C$ in $\mathbb{R}^2$ or $\mathbb{R}^3$, i.e., $\vec{r}'(t)$ is continuous and $\vec{r}''(t) \neq 0$, and that $\vec{r}(t)$ traverses $C$ exactly once.

Let $s$ be the arc length function for $\vec{r}(t)$, i.e., $ds/dt = \|\vec{r}'(t)\|$ and $s(a) = 0$.

The line integral of a function $f$ continuous on the smooth curve $C$ with respect to arc length is

$$\int_C f \, ds = \int_a^b f(\vec{r}(t))\|\vec{r}'(t)\| \, dt.$$ 

The value of the line integral does not depend on the parameterization $\vec{r}(t)$ of $C$ chosen as along as $\vec{r}(t)$ is smooth and traverses $C$ exactly once.

When $f = 1$ along $C$, the line integral gives the arc length of $C$.

Now suppose that $C$ is a piecewise smooth curve, i.e., it is the union of a finite number of smooth curves $C_1, \ldots, C_n$, where the terminal point of $C_i$ is the starting point of $C_{i+1}$.

The line integral of a function $f$ continuous on the piecewise smooth curve $C$ with respect to arc length is

$$\int_C f \, ds = \int_{C_1} f \, ds + \cdots + \int_{C_n} f \, ds.$$ 

Example. Let $f(x, y, z) = x + y + z$ and let $C$ be the union of the straight line segments $C_1$ from $(-1, 5, 0)$ to $(1, 6, 4)$ and $C_2$ from $(1, 6, 4)$ to $(0, 1, 1)$.

We parameterize a line segment from $\vec{a}$ to $\vec{b}$ in the standard way of

$$\vec{r}(t) = (1 - t)\vec{a} + t\vec{b}, \quad 0 \leq t \leq 1.$$ 

Thus a smooth parameterization of $C_1$ is

$$\vec{r}_1(t) = (1 - t)(-1, 5, 0) + t(1, 6, 4) = (2t - 1, t + 5, 4t), \quad 0 \leq t \leq 1,$n

and a smooth parameterization of $C_2$ is

$$\vec{r}_2(t) = (1 - t)(1, 6, 4) + t(0, 1, 1) = (-t + 1, -5t + 6, -3t + 4), \quad 0 \leq t \leq 1.$$ 

For these we have

$$\vec{r}_1''(t) = (2, 1, 4), \quad \vec{r}_2''(t) = (-1, -5, -3),$$

and so

$$\|\vec{r}_1''(t)\| = \sqrt{21}, \quad \|\vec{r}_2''(t)\| = \sqrt{35}.$$
The line integral of \( f \) along \( C \) with respect to arc length is
\[
\int_C f \, ds = \int_{C_1} f \, ds + \int_{C_2} f \, ds
\]
\[
= \int_0^1 (7t + 4)\sqrt{21} \, dt + \int_0^1 (-9t + 11)\sqrt{35} \, dt
\]
\[
= \sqrt{21} \left[ \frac{7t^2}{2} + 4t \right]_0^1 + \sqrt{35} \left[ -\frac{9t^2}{2} + 11t \right]_0^1
\]
\[
= \frac{15\sqrt{21} + 13\sqrt{35}}{2}
\]

Outcome B: Find the mass and center of mass of a thin wire given its shape and linear density.

For a thin wire in the shape of a curve \( C \) in the \( xy \)-plane and linear density \( \rho(x, y) \) on \( C \), the mass \( m \) of the thin wire is the line integral of the linear density along \( C \) with respect to arc length:
\[
m = \int_C \rho(x, y) \, ds.
\]
The center of mass \((\bar{x}, \bar{y})\) of such a thin wire has components
\[
\bar{x} = \frac{1}{m} \int_C x \rho(x, y) \, ds, \quad \bar{y} = \frac{1}{m} \int_C y \rho(x, y) \, ds.
\]
[The mass and center of mass extends readily to thin wires in \( \mathbb{R}^3 \).]

Example. A thin wire is in the shape of \( y = \sqrt{4 - x^2}, 0 \leq x \leq 2 \), with a linear density of \( \rho(x, y) = x \).

A standard parameterization of \( x^2 + y^2 = 4, 0 \leq x \leq 2 \), is \( \vec{r}(t) = \langle 2\cos t, 2\sin t \rangle \), \( 0 \leq t \leq \pi/2 \), where \( \|\vec{r}'(t)\| = 2 \).

The mass of this wire is
\[
m = \int_C \rho \, ds = 4 \int_0^{\pi/2} \cos t \, dt = 4 \left[ \sin t \right]_0^{\pi/2} = 4.
\]
The components of the center of mass are
\[
\bar{x} = \frac{1}{m} \int_C x \rho(x, y) \, ds = \frac{8}{4} \int_0^{\pi/2} \cos^2 t \, dt = 2 \left[ \frac{t}{2} + \frac{\sin 2t}{4} \right]_0^{\pi/2} = \frac{\pi}{2},
\]
and
\[
\bar{y} = \frac{1}{m} \int_C y \rho(x, y) \, ds = \frac{8}{4} \int_0^{\pi/2} \sin t \cos t \, dt = 2 \left[ \frac{\sin^2 t}{2} \right]_0^{\pi/2} = 1.
\]

Outcome C: Evaluate the line integral of a function along piecewise smooth curve with respect to \( x, y, \) or \( z \).

More line integrals of \( f \) are obtained by replacing \( ds \) with other differentials.
Let $C$ be a smooth curve parameterized by $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$, $a \leq t \leq b$.

The line integrals of functions $P$, $Q$, and $R$ along $C$ with respect to $x$, $y$, and $z$ are respectively

$$\int_C P \, dx = \int_a^b P(\vec{r}(t)) x'(t) \, dt,$$
$$\int_C Q \, dy = \int_a^b Q(\vec{r}(t)) y'(t) \, dt,$$
$$\int_C R \, dz = \int_a^b R(\vec{r}(t)) z'(t) \, dt.$$

These can be combined into one line integral as

$$\int_C P \, dx + Q \, dy + R \, dz = \int_a^b \left( P(\vec{r}(t)) x'(t) + Q(\vec{r}(t)) y'(t) + R(\vec{r}(t)) z'(t) \right) \, dt.$$

When there are only two variables $x$ and $y$, i.e., when $\vec{r}(t) = \langle x(t), y(t) \rangle$ and $P$ and $Q$ are just functions of $x$ and $y$, this line integral becomes

$$\int_C P \, dx + Q \, dy = \int_a^b \left( P(\vec{r}(t)) x'(t) + Q(\vec{r}(t)) y'(t) \right) \, dt.$$

The choice of parameterization $\vec{r}(t)$, $a \leq t \leq b$, of $C$ determines an orientation of $C$, i.e., the direction of motion along $C$ corresponding to increasing the value of $t$.

Reparameterizing the curve $C$ by $\vec{r}(b + a - t)$ gives the opposite orientation of $C$, and this oppositely orientated curve is denoted by $-C$.

Because switching the orientation of a curve changes the signs of $x'(t)$, $y'(t)$, and $z'(t)$, the relationship of the line integrals over $C$ and $-C$ is

$$\int_{-C} P \, dx + Q \, dy + R \, dz = - \int_C P \, dx + Q \, dy + R \, dz.$$

This does not happen with line integrals with respect to arc length, because $ds$ does not change sign with the opposite orientation.

For a piecewise smooth curve $C$, we add up the line integrals with respect to $x$, $y$, and $z$ over each smooth component $C_i$, ensuring that we have not inadvertently switched the orientation of $C_i$ by our choice of its parameterization.

Example. Let $C$ be the piecewise smooth boundary of the region $\{(x, y) : 0 \leq x, y \leq 1\}$, with a counterclockwise orientation.

Let $C_1$ be the bottom line segment; a parameterization for $C_1$ is $\vec{r}(t) = \langle t, 0 \rangle$, $0 \leq t \leq 1$.

Let $C_2$ be the right line segment; a parameterization of $C_2$ is $\vec{r}(t) = \langle 1, t \rangle$, $0 \leq t \leq 1$.

Let $C_3$ be the top line segment; a parameterization of $C_3$ is $\vec{r}(t) = \langle t, 1 \rangle$, $0 \leq t \leq 1$.

Let $C_4$ be the left line segment; a parameterization of $C_4$ is $\vec{r}(t) = \langle 0, t \rangle$, $0 \leq t \leq 1$. 
Are all of these line segments oriented correctly? $C_1$ and $C_2$ are orientated counterclockwise, but $C_3$ and $C_4$ are oriented clockwise.

And so we write $C = C_1 + C_2 - C_3 - C_4$.

For $P(x,y) = -y/2$ and $Q(x,y) = x/2$, the line integral over $C$ is

$$\int_C Pdx + Qdy = \int_{C_1} Pdx + Qdy + \int_{C_2} Pdx + Qdy - \int_{C_3} Pdx + Qdy - \int_{C_4} Pdx + Qdy$$

$$= \int_0^1 (0)(1dt) + (t/2)(0dt) + \int_0^1 (-t/2)(0dt) + (1/2)(1dt)$$

$$- \int_0^1 (-1/2)(1dt) + (t/2)(0dt) - \int_0^1 (-t/2)(0dt) + (0)(1dt)$$

$$= 0 + 1/2 - (-1/2) + 0$$

$$= 1.$$  

Curious that this line integral is equal to the area of the region which $C$ bounds. (More on this later in the week!)

**Outcome D**: Evaluate the line integral of a vector field along a piecewise smooth curve.

The **line integral** of a continuous vector field $\vec{F}$ along a smooth curve $C$ parameterized by $\vec{r}(t)$, $a \leq t \leq b$ is

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt.$$  

When we write $\vec{F} = \langle P, Q, R \rangle$ and $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$, we have

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \langle P(\vec{r}(t)), Q(\vec{r}(t)), R(\vec{r}(t)) \rangle \cdot \langle x'(t), y'(t), z'(t) \rangle \, dt$$

$$= \int_a^b (P(\vec{r}(t))x'(t) + Q(\vec{r}(t))y'(t) + R(\vec{r}(t))z'(t)) \, dt$$

$$= \int_C Pdx + Qdy + Rdz.$$  

So evaluating the line integral of $\vec{F}$ along smooth or piecewise smooth $C$ is done as we saw previously.

Since $ds/dt = \|\vec{r}'(t)\|$ and $\vec{T}(t) = \vec{r}'(t)/\|\vec{r}'(t)\|$ (the unit tangent vector), we also have

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \vec{T} \, ds.$$  

The line integral of $\vec{F}$ along $C$ represents the **work done** by the tangential component of the force $\vec{F}$ along $C$.  