Stoke’s Theorem is a higher dimensional version of Green’s Theorem.

Suppose $S$ is an oriented piecewise smooth surface with orientation $\vec{n}$ and a piecewise smooth simple closed boundary curve $C = \partial S$.

The positive orientation of $C$ (relative to the orientation on $S$) is given by the “right-hand rule,” i.e., when the fingers on your right hand move along $C$, your right-hand thumb points in the direction of $\vec{n}$. Here is rendering of this scenario.

Stoke’s Theorem. Let $S$ be an oriented piecewise smooth surface with a simple, closed, piecewise smooth boundary curve $C = \partial S$ with positive orientation. If $\vec{F}$ is a vector field whose components have continuous first-order partials on an open region $\mathbb{R}^3$ containing $S$, then

$$ \oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl} \vec{F} \cdot d\vec{S}. $$

Remark. Stoke’s Theorem says that the flux of the curl of $\vec{F}$ across $S$ is the same as the work done by $\vec{F}$ along $C$.

Proof (in a simple case). Suppose $S$ is the graph of $z = f(x, y)$, $(x, y) \in D$, where $f$ has continuous second-order partials, and assume the upward orientation for $S$.

The boundary $C = \partial S$ projects onto the $xy$-plane as the boundary $C_1 = \partial D$.

If $(x(t), y(t))$, $a \leq t \leq b$, is a “positive” parameterization of $C_1$, then

$$ \vec{r}(t) = (x(t), y(t), f(x(t), y(t))), \quad a \leq t \leq b, $$

is a positively oriented parameterization of $C$.

Let $\vec{F} = (P, Q, R)$ with components having continuous first-order partials.
Then the line integral of $\vec{F}$ over $C$ is
\[
\oint_{C} \vec{F} \cdot d\vec{r} = \int_{a}^{b} \left( P \frac{dx}{dt} + Q \frac{dy}{dt} + R \frac{dz}{dt} \right) dt
\]
\[
= \int_{a}^{b} \left[ \left( P + R \frac{\partial z}{\partial x} \right) \frac{dx}{dt} + \left( Q + R \frac{\partial z}{\partial y} \right) \frac{dy}{dt} \right] dt \quad \text{[by chain rule]}
\]
\[
= \oint_{C_1} \left( P + R \frac{\partial z}{\partial x} \right) dx + \left( Q + R \frac{\partial z}{\partial y} \right) dy
\]
\[
= \iint_{D} \left[ \frac{\partial}{\partial x} \left( Q + R \frac{\partial z}{\partial y} \right) - \frac{\partial}{\partial y} \left( P + R \frac{\partial z}{\partial x} \right) \right] dA \quad \text{[by Green's Theorem]}
\]
\[
= \iint_{D} \left[ \frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial z} \frac{\partial z}{\partial x} + \frac{\partial R}{\partial z} \frac{\partial z}{\partial x} + \frac{\partial R}{\partial z} \frac{\partial z}{\partial y} + \frac{\partial^2 z}{\partial x \partial y} \right] dA
\]
\[
= \iint_{D} \left[ - \left( \frac{\partial P}{\partial y} + \frac{\partial P}{\partial z} \frac{\partial z}{\partial y} + \frac{\partial R}{\partial z} \frac{\partial z}{\partial y} + \frac{\partial R}{\partial z} \frac{\partial z}{\partial x} + \frac{\partial^2 z}{\partial y \partial x} \right) \right] dA
\]
\[
= \iint_{D} \left[ - \left( \frac{\partial Q}{\partial y} + \frac{\partial Q}{\partial z} \frac{\partial z}{\partial y} - \frac{\partial P}{\partial z} - \frac{\partial R}{\partial z} \frac{\partial z}{\partial y} \right) \frac{\partial z}{\partial x} + \frac{\partial Q}{\partial z} \frac{\partial z}{\partial y} \frac{\partial z}{\partial x} \right] dA
\]
\[
= \iint_{D} \text{curl}(P, Q, R) \cdot \langle -z_x, -z_y, 1 \rangle \ dA
\]
\[
= \iint_{S} \text{curl} \vec{F} \cdot d\vec{S},
\]
where we have used Clairaut’s Theorem, and the given orientation for $S$. \hfill \Box

Outcome A: Apply Stoke’s Theorem to compute the flux of the curl of a vector field over an oriented surface.

Stoke’s Theorem is a \textit{computational tool} for the evaluation of the flux integral of the curl of a vector field when the boundary curve $C$ is easily parameterized.

Example. Let $\vec{F} = \langle xz, yz, xy \rangle$ and $S$ is the part of upper hemisphere $z = \sqrt{4 - x^2 - y^2}$ that lies within $x^2 + y^2 = 1$, oriented upwards.

The curl of the vector field is
\[
\text{curl} \vec{F} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
xz & yz & xy
\end{vmatrix} = \langle x - y, x - y, 0 \rangle.
\]

The boundary of $S$ is the curve of intersection between $z = \sqrt{4 - x^2 - y^2}$ and $x^2 + y^2 = 1$, i.e., $z = \sqrt{3}$ and $x^2 + y^2 = 1$.

A positively oriented parameterization for $C$ is
\[
\vec{r}(t) = \langle \cos t, \sin t, \sqrt{3} \rangle, \quad 0 \leq t \leq 2\pi.
\]
By Stoke’s Theorem,
\[ \int \int_S \text{curl} \, \vec{F} \cdot d\vec{S} = \oint_C \vec{F} \cdot d\vec{r} \]
\[ = \int_0^{2\pi} \langle \sqrt{3} \cos t, \sqrt{3} \sin t, \cos t \sin t \rangle \cdot \langle -\sin t, \cos t, 0 \rangle \, dt \]
\[ = \int_0^{2\pi} ( -\sqrt{3} \sin t \cos t + \sqrt{3} \sin t \cos t ) \, dt \]
\[ = 0. \]

Outcome B: Apply Stoke’s Theorem to compute the line integral of a vector field over an oriented closed curve.

Stoke’s Theorem is also a computational tool for the evaluation of a line integral of a vector field over a simple closed curve when the surface is easily parameterized.

Example. Let \( \vec{F} = \langle xy, 2z, 3y \rangle \), and \( C \) the intersection of \( x + z = 5 \) and \( x^2 + y^2 = 9 \).

The curl of the vector field is
\[ \text{curl} \, \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & 2z & 3y \end{vmatrix} = \langle 1, 0, -x \rangle. \]

The curve \( C \) lies in the plane \( z = 5 - x \), and so \( C \) is the boundary of the surface \( S : \vec{r}(x, y) = \langle x, y, 5 - x \rangle, (x, y) \in D \), where \( D \) is the circle with center at the origin in the \( xy \)-plane and radius 3.

An upward normal to this surface is
\[ \vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{vmatrix} = \langle 1, 0, 1 \rangle, \]
which is the vector normal to the plane \( x + z = 5 \).

By Stoke’s Theorem,
\[ \oint_C \vec{F} \cdot d\vec{r} = \int \int_S \text{curl} \, \vec{F} \cdot d\vec{S} \]
\[ = \int \int_D \langle 1, 0, -x \rangle \cdot \langle 1, 0, 1 \rangle \, dA = \int \int_D (1 - x) \, dA \]
\[ = \int_0^{2\pi} \int_0^3 (1 - r \cos \theta) r \, dr \, d\theta \]
\[ = 2\pi \int_0^3 r \, dr = 9\pi. \]