For finite sums we have the commutative and associative properties holding, but what about infinite sums?

Example. For positive integers \( i \) and \( j \), consider the numbers

\[
a_{ij} = \begin{cases} 
2^{i-j} & \text{if } j > i, \\
-1 & \text{if } i = j, \\
0 & \text{if } j < i.
\end{cases}
\]

We can visualize these numbers by a grid:

\[
\begin{bmatrix}
-1 & 1/2 & 1/4 & 1/8 & 1/16 & \\
0 & -1 & 1/2 & 1/4 & 1/8 & \\
0 & 0 & -1 & 1/2 & 1/4 & \\
0 & 0 & 0 & -1 & 1/2 & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
\end{bmatrix}
\]

What happens when we add the rows first? We get

\[
\sum_{i=1}^{\infty} \left( \sum_{j=1}^{\infty} a_{ij} \right) = \sum_{i=1}^{\infty} 0 = 0
\]

because \( 1/2 + 1/4 + 1/8 + \cdots = 1 \) (geometric series for \( r = 1/2 \) with first term 1 missing).

What happens when we add the columns first? We get

\[
\sum_{j=1}^{\infty} \left( \sum_{i=1}^{\infty} a_{ij} \right) = -1 - 1/2 - 1/4 - 1/8 - \cdots = -2
\]

because \( 1 + 1/2 + 1/4 + 1/8 + \cdots = 2 \) (geometric series for \( r = 1/2 \)).

This shows that commutativity of addition can fail for infinite sums.

Example. Consider the series

\[
\sum_{n=1}^{\infty} (-1)^n = -1 + 1 - 1 + 1 - 1 + 1 - 1 + \cdots.
\]

Associating the sum in one way gives

\[
(-1 + 1) + (-1 + 1) + (-1 + 1) + \cdots = 0 + 0 + 0 + \cdots = 0,
\]

while associating the sum in another way gives

\[
-1 + (1 - 1) + (1 - 1) + (1 - 1) + \cdots = -1 + 0 + 0 + \cdots = -1.
\]
This shows that associativity of addition can fail for infinite sums.
An understanding of series depends heavily upon an understanding of sequences and their convergence or divergence.

Definition 2.2.1. A **sequence** is a function whose domain is $\mathbb{N}$.
We typically write a sequence as $(a_n)_{n=1}^{\infty}$, or simply $(a_n)$ where $n \in \mathbb{N}$ is implicitly understood.

Examples.

(a) $(2^{-n+1})_{n=1}^{\infty} = \left(1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots \right)$.
(b) $\left(\frac{1+n}{n}\right) = \left(2, \frac{3}{2}, \frac{4}{3}, \cdots \right)$.

Definition 2.2.3. A sequence $(a_n)$ **converges** to a real number $a$ if, for every positive $\epsilon > 0$, there exists a positive integer $N$ such that whenever $n \geq N$ it follows that $|a_n - a| < \epsilon$.

Some notations for a convergence sequence $(a_n)$ are

$$\lim_{n \to \infty} a_n = a, \quad \lim a_n = a, \quad (a_n) \to a.$$ 

The notion of $|a_n - a| < \epsilon$ requires some special attention.

Definition 2.2.4. Given $a \in \mathbb{R}$ and a positive $\epsilon > 0$, the set

$$V_\epsilon(a) = \{x \in \mathbb{R} : |x - a| < \epsilon\}$$

is called the $\epsilon$-neighbourhood of $a$.
This set $V_\epsilon(a)$ is an open interval with $a$ at its center and with a "radius" of $\epsilon$.
This notion of a neighbourhood leads to a "topological" version of convergence.

Definition 2.2.3B. A sequence $(a_n)$ converges to $a$ if, given any $\epsilon$-neighbourhood $V_\epsilon(a)$, there exists a point (a.k.a. $N$) in the sequence after which all of the terms of the sequence are in $V_\epsilon(a)$.
This says that only finitely many terms of the sequence are not in $V_\epsilon(a)$.
The number $N$ is the point where the sequence enters $V_\epsilon(a)$ and never leaves.
You should recognize that the value of $N$ will generally depend on the choice of $\epsilon$: the smaller $\epsilon$, the bigger the value of $N$ for which the sequence enters $V_\epsilon(a)$ never to leave.
Usually the choice of $N$ can be determined by how the terms in the sequence $(a_n)$ are defined by $n$.

Example 2.2.5. Consider $(a_n)$ for $a_n = 1/\sqrt{n}$. 

As \( n \) gets bigger (i.e., approaches \( \infty \)), the value of \( a_n \) approaches 0, and we “conclude” that
\[
\lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0.
\]
To prove this rigourously we need to understand the relationship between the choice of \( \epsilon \) and the value of \( N \) needed to have \( a_n \in V_\epsilon(0) \) for all \( n \geq N \).

If we take \( \epsilon = 1/10 \), then we are seeking for a value of \( N \) such that
\[
\left| \frac{1}{\sqrt{n}} - 0 \right| < \frac{1}{10}.
\]
We recognize that with \( n = 100 \) we get \( 1/\sqrt{n} = 1/10 \), and so we can pick \( N = 101 \) or any larger integer.

If we take \( \epsilon = 1/50 \), then we are seeking for a value of \( N \) such that
\[
\left| \frac{1}{\sqrt{n}} - 0 \right| < \frac{1}{50}.
\]
That is we are solving
\[
\frac{1}{\sqrt{n}} < \frac{1}{50}
\]
for \( n \) which gives
\[
n > 50^2 = 2500.
\]
We can pick \( N = 2501 \) or any larger integer.

The whole point of this is that no matter small we choose \( \epsilon \) to be, we can find a value of \( N \) for which
\[
\left| \frac{1}{\sqrt{n}} - 0 \right| < \epsilon
\]
for all \( n \geq N \):
\[
\frac{1}{\sqrt{n}} < \epsilon \Rightarrow n > \frac{1}{\epsilon^2}.
\]
We then have a proof that the sequence converges to 0: for every \( \epsilon > 0 \) choose \( N \in \mathbb{N} \) by
\[
N > \frac{1}{\epsilon^2}.
\]
Then for all \( n \geq N \), we have
\[
\left| \frac{1}{\sqrt{n}} - 0 \right| = \frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{N}} < \epsilon.
\]
The first inequality follows because \( n \geq N \), and the second inequality follows because \( N > 1/\epsilon^2 \).

Not all sequences converge, like \((a_n) = (-1)^n = (-1, 1, -1, 1, -1, \cdots)\). Why does this not converge?

Definition 2.2.9. A sequence that does not converge is said to diverge.