Math 341 Lecture #10
§2.5: Subsequences and The Bolzano-Weierstrass Theorem

Definition 2.5.1. Let (a_n) be a sequence of real numbers, and let $n_1 < n_2 < n_3 < \cdots$ be a strictly increasing sequence of natural numbers. Then the sequence

$$a_{n_1}, a_{n_2}, a_{n_3}, \ldots,$$

is called a *subsequence* of (a_n) and is denoted by (a_{n_j}), where $j \in \mathbb{N}$ indexes the subsequence.

Notice that the order of the terms in a subsequence (a_{n_j}) is the same as in the original sequence (a_n).

Example. If $a_n = 1/n^2$, then

$$\left(1, \frac{1}{9}, \frac{1}{25}, \frac{1}{49}, \ldots\right)$$

and

$$\left(\frac{1}{4}, \frac{1}{16}, \frac{1}{64}, \frac{1}{256}, \ldots\right)$$

are subsequences of (a_n).

For the first of these we have (a_{n_j}) where

$$n_j = 2j - 1$$

and for the second of these we have

$$n_j = 2^j.$$

Both of these forms of n_j give strictly increasing sequences of positive integers.

Theorem 2.5.2. All subsequences of a convergent sequence converge to the same limit as the original sequence.

Proof. Let (a_n) be a convergence sequence with limit l.

Suppose (a_{n_j}) is a subsequence of (a_n).

For $\epsilon > 0$ we must find $J \in \mathbb{N}$ such that $|a_{n_j} - l| < \epsilon$ for all $j \geq J$.

Since $a_n \to l$ as $n \to \infty$, there is $N \in \mathbb{N}$ such that $|a_n - l| < \epsilon$ for all $n \geq N$.

By the nature of n_j, there is a $J \in \mathbb{N}$ such that $n_j \geq N$ for all $j \geq J$.

Then because $|a_n - l| < \epsilon$ for all $n \geq N$, and because $n_j \geq N$ for all $j \geq J$, we have that $|a_{n_j} - l| < \epsilon$ for all $j \geq J$. □

Example 2.5.3. For $0 < b < 1$ we have

$$b > b^2 > b^3 > b^4 > \cdots > b^n > \cdots > 0.$$

Thus the sequence (b^n) is decreasing and bounded below, and so it converges by the Monotone Convergence Theorem.

A reasonable guess for the limit is 0, but we can confirm that by the Algebraic Limit Theorem and a strategic choice of a subsequence.

If l is the limit of (b^n), then l is the limit of the subsequence (b^{2n}).

But $b^{2n} = b^n b^n$ and so we have $l = l^2$, and thus $l = 0$ (why not 1?).

Can you extend this to $-1 < b < 0$? It is true.

Divergence Criterion for Sequences 2.5.4. Since all subsequences of a convergence sequence converge to the same limit as the original, then we can detect a divergence sequence if we can produce two subsequences that converge to different limits.

The sequence $(-1)^n$ is not convergent because it has two subsequences $(-1)^{2n}$ and $(-1)^{2n+1}$ which converge to 1 and -1 respectively.

Recall that a convergence sequence is bounded, but that a bounded sequence is not necessarily convergent: think about about $(-1)^n$.

But as we have seen, a bounded sequence might have a convergent subsequence, like $(-1)^n$ does.

It is an amazing result that every bounded sequence has a convergent subsequence.

The Bolzano-Weierstrass Theorem 2.5.5. Every bounded sequence contains a convergent subsequence.

Proof. Let (a_n) be a bounded sequence.

Then there is $M \in \mathbb{R}$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$.

We will construct a convergent subsequence of (a_n) through a bisection technique.

Bisect the closed interval $[-M, M]$ into the closed subintervals $[-M, 0]$ and $[0, M]$.

Notice the midpoint is included in both subintervals, but as we shall see, this does not complicate things.

Since there are infinitely many a_n, one of the two subintervals must contain infinitely many of them; label this closed interval I_1 and choose n_1 so that $a_{n_1} \in I_1$.

Now bisect the closed interval I_1 into two closed subintervals that overlap at the midpoint.

Since there are infinitely many a_n for $n > n_1$, one of the two closed subintervals must contain infinitely many of them; label this closed interval I_2, and choose $n_2 > n_1$ so that $a_{n_2} \in I_2$.

Notice that $I_1 \supseteq I_2$.

We can repeat this step countably many times to obtain a nested sequence of closed intervals

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq \cdots$$

and positive integers $n_1 < n_2 < n_3 < n_4 < \cdots$ such that $a_{n_j} \in I_j$ for all $j \in \mathbb{N}$.

By the Nested Interval Property (Theorem 1.4.1) there is at least one $x \in \mathbb{R}$ contained in every I_j.

Now the suspicion is that this x is the limit of the subsequence (a_{n_j}).

Let $\epsilon > 0$.

By the bisection technique, the length of I_j is $M(1/2)^{j-1}$ which converges to 0.

Choose J so that $j \geq J$ implies that the length of I_j is less than ϵ.

Then as a_{n_j} and x are both in the closed interval I_j of length less than ϵ, we have

$$|a_{n_j} - x| < \epsilon$$

for all $j \geq J$.

This holds for all $j \geq J$ because of the nested property of I_j and because $a_{n_j} \in I_j$.

Thus we have that (a_{n_j}) converges to x. $\square$