Math 341 Lecture #12

§2.7: Infinite Series

Recall that the convergence of an infinite series \( \sum_{k=1}^{\infty} a_k \) is defined in terms of the convergence of the sequence of its partial sums \( (s_m) \), where \( s_m = \sum_{k=1}^{m} a_k \):

\[
\sum_{k=1}^{\infty} a_k = A \quad \text{means that} \quad \lim_{m \to \infty} s_m = A.
\]

Thus we can translate results about convergent sequences to convergent series.

Theorem 2.7.1 (Algebraic Limit Theorem for Series). If \( \sum_{k=1}^{\infty} a_k = A \) and \( \sum_{k=1}^{\infty} b_k = B \), then

(a) \( \sum_{k=1}^{\infty} c a_k = cA \) for all \( c \in \mathbb{R} \), and

(b) \( \sum_{k=1}^{\infty} (a_k + b_k) = A + B \).

The proof of this is in the Appendix.

Missing from Theorem 2.7.1 is any mention about the product of two series; see Section 2.8 for this.

Theorem 2.7.2 (Cauchy Criterion for Series). The series \( \sum_{k=1}^{\infty} a_k \) converges if and only if, for given \( \epsilon > 0 \), there exists an \( N \in \mathbb{N} \) such that whenever \( n > m \geq N \) it follows that

\[ |a_{m+1} + a_{m+2} + \cdots + a_n| < \epsilon. \]

Proof. Observe for positive integers \( n > m \) that

\[ |s_n - s_m| = |a_{m+1} + a_{m+2} + \cdots + a_n|. \]

Now apply the Cauchy Criterion for sequences. \( \square \)

Theorem 2.7.3. If \( \sum_{k=1}^{\infty} a_k \) converges, then \( (a_k) \to 0 \).

The proof of this is in the Appendix.

The converse of Theorem 2.7.3 is false because of the Harmonic series.

The contrapositive of Theorem 2.7.3 – if \( (a_k) \not\to 0 \), then \( \sum_{k=1}^{\infty} a_k \) diverges – gives a divergence test for series.

Theorem 2.7.4 (Comparison Test). Assume \( (a_k) \) and \( (b_k) \) satisfy \( 0 \leq a_k \leq b_k \) for all \( k \in \mathbb{N} \).

(a) If \( \sum_{k=1}^{\infty} b_k \) converges, then \( \sum_{k=1}^{\infty} a_k \) converges.

(b) If \( \sum_{k=1}^{\infty} a_k \) diverges, then \( \sum_{k=1}^{\infty} b_k \) diverges.
The proof of this is in the Appendix.
The usefulness of the Comparison Test depends on knowing series that converge or diverge.
Recall that we know $\sum_{n=1}^{\infty} 1/n^p$ converges if and only if $p > 1$.

**Example 2.7.5 (Geometric Series).** When $|r| < 1$, we have that

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1 - r}.$$ 

**Theorem 2.7.6 (Absolute Convergence Test).** If the series $\sum_{k=1}^{\infty} |a_k|$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.

Proof. Suppose $\sum_{k=1}^{\infty} |a_k|$ converges.

By the Cauchy Criterion for Series, for $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that

$$|a_{m+1}| + |a_{m+2}| + \cdots + |a_n| = |a_{m+1}| + |a_{m+2}| + \cdots + |a_n| < \epsilon$$

for all $n > m \geq N$.

By the triangle inequality, we have

$$|a_{m+1} + a_{m+2} + \cdots + a_n| \leq |a_{m+1}| + |a_{m+2}| + \cdots + |a_n|,$$

for all $n > m \geq N$, and so by the Cauchy Criterion for Series, we have that $\sum_{k=1}^{\infty} a_k$ converges.

The converse of the Absolute Convergence Test is false, as the alternating series

$$\sum_{k=1}^{\infty} (-1)^{k+1}/k$$
demonstrates.

**Theorem 2.7.7 (Alternating Series Test).** Let $(a_n)$ be a sequence satisfying

(i) $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq a_{n+1} \geq \cdots$, and

(ii) $(a_n) \to 0$.

Then the alternating series $\sum_{k=1}^{\infty} (-1)^{k+1}a_k$ converges.

The proof of this is a homework problem 2.7.1.

**Definition 2.7.8.** We say a series $\sum_{k=1}^{\infty} a_k$ converges absolutely if $\sum_{k=1}^{\infty} |a_k|$ converges.

We say a series $\sum_{k=1}^{\infty} a_k$ converges conditionally if $\sum_{k=1}^{\infty} a_k$ converges but $\sum_{k=1}^{\infty} |a_k|$ diverges.

We can now address the issue of the order of addition in an infinite series.
Definition 2.7.9. A series $\sum_{k=1}^{\infty} b_k$ is a rearrangement of a series $\sum_{k=1}^{\infty} a_k$ if there is a bijection $f : \mathbb{N} \to \mathbb{N}$ such that $b_{f(k)} = a_k$ for all $k \in \mathbb{N}$.

Theorem 2.7.10. If $\sum_{k=1}^{\infty} a_k$ converges absolutely, then any rearrangement of $\sum_{k=1}^{\infty} a_k$ converges to the same limit.

Proof. Assume $\sum_{k=1}^{\infty} a_k$ converges absolutely to $A$, and let $\sum_{k=1}^{\infty} b_k$ be a rearrangement of $\sum_{k=1}^{\infty} a_k$.

For the partial sums set

$$s_m = a_1 + \cdots + a_m, \quad t_m = b_1 + \cdots + b_m.$$  

We want to show that $(t_m) \to A$.

By the convergence of $\sum_{k=1}^{\infty} a_k$, for $\epsilon > 0$, there exists an $N_1 \in \mathbb{N}$ such that

$$|s_m - A| < \frac{\epsilon}{2} \quad \text{for all} \quad m \geq N_1.$$

Applying the Cauchy Criterion to the convergent $\sum_{k=1}^{\infty} |a_k|$, there is an $N_2 \in \mathbb{N}$ such that

$$\sum_{k=m+1}^{n} |a_k| < \frac{\epsilon}{2} \quad \text{for all} \quad n > m \geq N_2.$$

Now we take $N = \max\{N_1, N_2\}$.

The finite set of terms $\{a_1, a_2, \ldots, a_N\}$ appears somewhere in the rearranged series $\sum_{k=1}^{\infty} b_k$.

If $f : \mathbb{N} \to \mathbb{N}$ is the bijection satisfying $b_{f(k)} = a_k$, then we can move far enough out in the series $\sum_{k=1}^{\infty} b_k$ to account for $\{a_1, \ldots, a_N\}$ by choosing

$$M = \max\{f(k) : 1 \leq k \leq N\}.$$  

This choice of $M$ satisfies $M \geq N$ because $f(k) \geq N$ for some $1 \leq k \leq N$.

For $m \geq M$, the difference $t_m - s_N$ consists of a finite number of the $a_k$ terms which, for large enough $n \geq N$, all appear in $\sum_{k=N+1}^{n} a_k$.

Let $g : \mathbb{N} \to \{0, 1\}$ be the function defined by $g(k) = 1$ if $a_k$ appears in $t_m - s_N$ and $g(k) = 0$ if $a_k$ does not appear in $t_m - s_N$.

Then

$$t_m - s_N = \sum_{k=N+1}^{n} g(k)a_k,$$

and so

$$|t_m - s_N| = \left| \sum_{k=N+1}^{n} g(k)a_k \right| \leq \sum_{k=N+1}^{n} |g(k)a_k| \leq \sum_{k=N+1}^{n} |a_k|.$$
The choice of $N_2$ guarantees that $|t_m - s_N| < \epsilon/2$ when $m \geq M \geq N_2$, and so

$$
|t_m - A| = |t_m - s_N + s_N - A| \\
\leq |t_m - s_N| + |s_N - A| \\
< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
= \epsilon
$$

whenever $m \geq M$, and thus $t_m$ converges to $A$. 

[□]
Appendix: Some Proofs

Proof of Theorem 2.7.1. (i) A partial sum of $\sum_{k=1}^{\infty} c a_k$ is

$$t_m = c a_1 + \cdots + c a_m.$$

A partial sum of $\sum_{k=1}^{\infty} a_k$ is

$$s_m = a_1 + \cdots + a_m.$$

Then $t_m = c s_m$, and since $s_m \to A$, we obtain by the Algebraic Limit Theorem (for sequences) that $t_m \to cA$.

(ii) A partial sum of $\sum_{k=1}^{\infty} (a_k + b_k)$ is

$$w_m = a_1 + b_1 + \cdots + (a_m + b_m) = a_1 + \cdots + a_m + b_1 + \cdots + b_m.$$

Partial sums for $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ are

$$s_m = a_1 + \cdots + a_m, \quad t_m = b_1 + \cdots + b_m.$$

Then $w_m = s_m + t_m$, and since $s_m \to A$ and $t_m \to B$, we have by the Algebraic Limit Theorem for sequences that $w_m \to A + B$.

Proof of Theorem 2.7.3. For a convergent series, the sequence of partial sums $(s_m)$ is Cauchy by Theorem 2.7.2.

Thus for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|s_n - s_m| < \epsilon$ whenever $n, m \geq N$. Choosing $n = m + 1$ gives $|s_{m+1} - s_m| < \epsilon$ for all $m \geq N$.

Here $s_{m+1} - s_m = a_{m+1}$, so we have $|a_{m+1}| < \epsilon$ for all $m \geq N$.

This says that $(a_m) \to 0$.

Proof of Theorem 2.7.4. Both of the comparisons follow immediately from the Cauchy Criterion for Series and the observation that

$$|a_{m+1} + \cdots + a_n| \leq |b_{m+1} + \cdots + b_n|$$

for $n > m$.

Proof of the convergence of the Geometric Series. A series is called geometric if it is of the form

$$\sum_{k=0}^{\infty} a r^k = a + ar + ar^2 + \cdots.$$

If $|r| \geq 1$ and $a \neq 0$, this series diverges because the terms do not go to zero.

For $r \neq 1$, the algebraic identity

$$(1 - r)(1 + r + r^2 + \cdots + r^{m-1}) = 1 - r^m$$

enables us to rewrite the partial sum term

$$s_m = a + ar + \cdots + ar^{m-1} = \frac{a(1 - r^m)}{1 - r}.$$
By the Algebraic Limit Theorem we have

\[
\lim_{m \to \infty} s_m = \lim_{m \to \infty} \frac{a(1 - r^m)}{1 - r} = \frac{a}{1 - r} \left( 1 - \lim_{m \to \infty} r^m \right),
\]

where the limit converges to 0 when \(|r| < 1\) (which we saw in Lecture 7).

Thus, when \(|r| < 1\) we conclude that

\[
\sum_{k=0}^{\infty} ar^k = \frac{a}{1 - r}.
\]