Some of you might remember that in Math 314 you may learned a little bit about compact sets, that they had something to do with being closed and bounded.

We will put all of this on a rigorous foundation now, using sequences.

**Definition 3.3.1.** A set $K \subseteq \mathbb{R}$ is *compact* if every sequence in $K$ has a subsequence that converges to a limit that is also in $K$.

**Example 3.3.2.** A closed interval $[c, d]$ with $-\infty < c < d < \infty$ is a compact set.

The Bolzano-Weierstrass Theorem and the Order Limit Theorem guarantee that any sequence $(a_n)$ with $c \leq a_n \leq d$ for all $n \in \mathbb{N}$ has a convergent subsequence $(a_{n_k})$ whose limit is in $[c, d]$.

The closed interval $[0, \infty)$ is not compact because the sequence $\{n\}$ in $[0, \infty)$ does not have a convergent subsequence.

What is the difference?

**Definition 3.3.3.** A set $A \subseteq \mathbb{R}$ is *bounded* if there exists $M > 0$ such that $|a| \leq M$ for all $a \in A$.

**Theorem 3.3.4.** A set $K \subseteq \mathbb{R}$ is compact if and only if it is closed and bounded.

Proof. Let $K$ be compact.

To show that $K$ is bounded, suppose that $K$ is unbounded.

Then for every $n \in \mathbb{N}$ there is $x_n \in K$ such that $|x_n| > n$.

Since $K$ is compact, the sequence $(x_n)$ has a convergent, hence bounded, subsequence $(x_{n_j})$.

But $|x_{n_j}| > n_j$ with $n_j \to \infty$ as $j \to \infty$, a contradiction.

So $K$ is bounded.

To see that $K$ is closed, we take a limit point $x$ of $K$ and a sequence $(x_n)$ with $x_n \in K$ and $x_n \neq x$ for all $n \in \mathbb{N}$ such that $(x_n) \to x$, and show that $x \in K$.

The compactness of $K$ implies that there is a subsequence $(x_{n_j})$ that converges to a point that is in $K$.

Since $(x_n) \to x$, then $(x_{n_j}) \to x$ as well, and so $x \in K$, and $K$ is closed.

Showing that a closed and bounded set is compact is a homework problem 3.3.3. \[ \square \]

We can replace the bounded and closed intervals in the Nested Interval Property with compact sets, and get the same result.

**Theorem 3.3.5.** If $K_1 \supseteq K_2 \supseteq K_3 \supseteq \cdots$ for compact sets $K_i \subseteq \mathbb{R}$, then $\cap_{i=1}^{\infty} K_i \neq \emptyset$.

Proof. For each $n \in \mathbb{N}$ pick $x_n \in K_n$.

Because the compact sets are nested, the sequence $(x_n)$ is contained in $K_1$.

Since $K_1$ is compact, there is a convergent subsequence $(x_{n_j})$ with limit $x \in K_1$. 

We will show that \( x \in K_n \) for all \( n \in \mathbb{Z} \), and hence \( x \in \bigcap_{n=1}^{\infty} K_n \).

For each \( n_0 \in \mathbb{N} \) we have by the nesting that \( x_n \in K_{n_0} \) for all \( n \geq n_0 \).

Because \( n_k \) is a strictly increasing function, there is a choice of \( k_0 \) such that for all \( k \geq k_0 \) we have \( x_{n_k} \in K_{n_0} \).

The compactness of \( K_{n_0} \) implies that \( x \in K_{n_0} \).

Since \( n_0 \) is arbitrary, we have that \( x \in K_n \) for all \( n \in \mathbb{N} \). \( \square \)

There is another equivalent way to describe (and define) compactness of sets by the use of open sets.

**Definition 3.3.6.** An open cover for \( A \subseteq \mathbb{R} \) is a (possibly infinite) collection of open sets \( \{O_\lambda : \lambda \in \Lambda\} \) for which

\[
A \subseteq \bigcup_{\lambda \in \Lambda} O_\lambda.
\]

For a given open cover \( \{O_\lambda : \lambda \in \Lambda\} \) of \( A \), a finite subcover is a finite subcollection of open sets \( O_{\lambda_1}, \ldots, O_{\lambda_k} \) in \( \{O_\lambda : \lambda \in \Lambda\} \) such that

\[
A \subseteq \bigcup_{i=1}^{k} O_{\lambda_i}.
\]

**Example 3.3.7.** For each \( x \in (0, 1) \), let \( O_x = (x/2, 1) \).

Then the collection \( \{O_x : x \in (0, 1)\} \) is an open cover of \( (0, 1) \) because each \( y \in (0, 1) \) belongs to \( O_x \) for an \( x \) satisfying \( 0 < x/2 < y \).

Does this open cover of \( (0, 1) \) have a finite subcover?

Suppose there is a finite subcover: there are \( x_1, \ldots, x_n \in (0, 1) \) such that \( O_{x_1}, \ldots, O_{x_n} \) is a finite subcover of \( (0, 1) \).

For \( x' = \min\{x_1, \ldots, x_n\} \), choose \( y \in (0, x'/2] \).

Since \( O_{x_1}, \ldots, O_{x_n} \) is a finite subcover of \( (0, 1) \), then

\[
y \in \bigcup_{k=1}^{n} O_{x_k}.
\]

But \( 0 < y \leq x_k/2 \) for all \( k = 1, \ldots, n \), so that \( y \not\in O_{x_k} \) for all \( k = 1, \ldots, n \), and hence

\[
y \not\in \bigcup_{k=1}^{n} O_{x_k}.
\]

Thus the cover \( \{O_x : x \in (0, 1)\} \) of \( (0, 1) \) does not have a finite subcover.

Is the set \( (0, 1) \) compact?

**Theorem 3.3.8 (Heine-Borel).** For \( K \subseteq \mathbb{R} \), the following are equivalent.

(i) \( K \) is compact.

(ii) \( K \) is closed and bounded.
(iii) Any open cover for $K$ has a finite subcover.

Proof. The equivalence of (i) and (ii) is Theorem 3.3.4.
We will show that (iii) implies (ii).

Suppose that (iii) holds: every open cover of $K$ has a finite subcover.
To show that $K$ is bounded, we consider the open cover $\{V_1(x) : x \in K\}$.
Notice that each $V_1(x)$ has a bounded length of 2.
This open cover has a finite cover: there exist finitely many elements $x_1, \ldots, x_k \in K$ such that
\[ K \subseteq V_1(x_1) \cup \cdots \cup V_1(x_k). \]
Because the finite cover consists of finitely many open intervals of length 2, the set $K$ must be bounded.

To show that $K$ is closed is more delicate, and it is obtained by contradiction.
Recall Theorem 3.2.8 which states that a set is closed if and only if every Cauchy sequence in the set has its limit in the set as well.
Let $(y_n)$ be a Cauchy sequence in $K$ whose limit $y \not\in K$.
Every $x \in K$ is a positive distance away from $y$, i.e., $\epsilon_x = |x - y|/2 > 0$ for all $x \in K$.
Let $O_x = V_{\epsilon_x}(x)$.
The open cover $\{O_x : x \in K\}$ of $K$ has a finite subcover $O_{x_1}, \ldots, O_{x_k}$.
Set $\epsilon_0 = \min\{\epsilon_{x_i} : i = 1, \ldots, k\}$.
Then $y$ is at least a distance of $2\epsilon_0$ away from each of $x_1, \ldots, x_k$.
Also for this $\epsilon_0$ there is $N \in \mathbb{N}$ such that $|y - y_N| < \epsilon_0$, that is, $y_N$ is within a distance of $\epsilon_0$ of $y$.
This implies that $y_N \not\in O_{x_i}$ for all $i = 1, \ldots, k$, and so $y_N \not\in \bigcup_{i=1}^{k} O_{x_i}$.
But $y_N \in K$ and so $y_N$ is in the finite subcover, a contradiction.
Thus $y \in K$, and $K$ is closed.
Showing that (ii) implies (iii) is left to you to consider (an outline is given in problem 3.3.9 which is not assigned as homework). $\square$