Math 341 Lecture #17
§3.4 Perfect Sets and Connected Sets.

The Cantor set $C$ has another topological property that will prove useful in showing that $C$ is uncountable.

Definition 3.4.1. A set $P \subset \mathbb{R}$ is perfect if it is closed and contains no isolated points.
A finite subset of $\mathbb{R}$ is closed but it is not perfect.
Closed intervals $[c, d]$ with $-\infty < c < d < \infty$, are perfect.
What about the Cantor set?

Theorem 3.4.2. The Cantor set $C$ is perfect.

Proof. Each $C_n$ is a finite union of closed intervals, and so is closed.
Then $C = \cap C_n$ is a closed set.
Now we will show that each $x \in C$ is not isolated by constructing a sequence $(x_n)$ in $C$ with $x_n \neq x$ for all $n \in \mathbb{N}$ and $x_n \to x$.
The closed set $C_1$ is the union of two closed intervals $I_{11}$ and $I_{12}$ each of length $1/3$.
The point $x$ is in one of these two closed intervals, call it $I_1$.
The intersection $C_2 \cap I_1$ consists of two closed intervals, one of which contains $x$.
Pick $x_1$ to be an endpoint of the other closed interval, and so $x_1 \neq x$.
Because the endpoints of the closed intervals that make $C_2$ are in $C$, we have that $x_1 \in C$.
Because $x$ and $x_1$ are both in $I_1$ and the length of $I_1$ is $1/3$, we have $|x_1 - x| \leq 1/3$.
The closed set $C_2$ is the union of four closed intervals $I_{2j}$, $j = 1, 2, 3, 4$, each of length $1/9$.
The point $x$ is in one of these four closed intervals, call it $I_2$.
The intersection $C_3 \cap I_2$ consists of two closed intervals, one of which contains $x$.
Pick $x_2$ to be an endpoint of the other closed interval, and so $x_2 \neq x$.
Because $x_2$ is an endpoint of one of the closed intervals in $C_3$, we have that $x_2 \in C$.
Because $x_2$ and $x$ both belong to $I_2$ which is of length $1/9$, we have $|x_2 - x| \leq 1/9$.
Continuing in this way, we construct a sequence $(x_n)$ in $C$ with $x_n \neq x$ for all $n \in \mathbb{N}$, and $|x_n - x| \leq 1/3^n$.
Thus we have shown that $x$ is a limit point of $C$, and therefore $x$ is not isolated.
As $x$ was an arbitrary point of $C$, we have that $C$ is perfect. \qed

In this proof we used the endpoints of the closed intervals in $C_n$ to form a sequence $(x_n)$ that converged to the given point $x$ in the Cantor set.
Each endpoint is rational of the form $m/3^n$ for $0 \leq m \leq 3^n$, but this does not mean that the limit $x$ of $(x_n)$ is rational.

In fact “most” of the sequences formed from the rational endpoints converge to irrational numbers, and this account for the uncountable nature of the Cantor set.
Theorem 3.4.3. A nonempty perfect set is uncountable.

Proof. A nonempty perfect set $P$ cannot be finite, because in a nonempty finite set each point is isolated.

So a nonempty perfect set is infinite.

Suppose, to the contrary, that $P$ is countable.

Using a bijection $f : \mathbb{N} \to P$ we can enumerate the elements of $P$ as

$$P = \{x_1, x_2, x_3, \ldots\}.$$

The point here is that every element of $P$ appears in this enumerated list.

Let $I_1$ be a closed interval that contains $x_1$ in its interior (i.e., $x_1$ is not an endpoint of $I_1$).

As $P$ is perfect, the element $x_1$ is not isolated, so there is some other $y_2 \in P$ such that $y_2$ is also in the interior of $I_1$.

Choose a closed interval $I_2$ centered on $y_2$ so that $I_2 \subseteq I_1$ and $x_1 \notin I_2$.

Since $y_2 \in I_2$ and $y_2 \in P$ we have $I_2 \cap P \neq \emptyset$.

The element $y_2 \in P$ is not isolated, so there is a $y_3 \in P$ that is in the interior of $I_2$.

We may choose $y_3 \neq x_2$, for if $y_3 = x_2$ then there will be another choice of $y_3 \in P$ in the interior of $I_2$ because $y_3$ is not isolated.

Now choose a closed interval $I_3 \subseteq I_2$ centered on $y_3$ for which $x_2 \notin I_3$.

Since $y_3 \in I_3$ and $y_3 \in P$ we have $I_3 \cap P \neq \emptyset$.

Carrying out this construction inductively results in a sequence of closed intervals $I_n$ satisfying

(i) $I_{n+1} \subseteq I_n$,
(ii) $x_n \notin I_{n+1}$, and
(iii) $I_n \cap P \neq \emptyset$.

For each $n \in \mathbb{N}$, the set $K_n = I_n \cap P$ is compact because $I_n$ is bounded and $I_n \cap P$ closed.

By Theorem 3.3.5 we have that

$$\bigcap_{n=1}^{\infty} K_n \neq \emptyset.$$

On the other hand, since $K_n \subseteq P$ and $x_n \notin I_{n+1}$ for all $n \in \mathbb{N}$, we have that

$$\bigcap_{n=1}^{\infty} K_n = \emptyset.$$

This contradiction implies that $P$ is uncountable.  \qed
Now we turn our attention to another topological notion for subsets of \( \mathbb{R} \).

**Definition 3.4.4.** Two nonempty sets \( A, B \subseteq \mathbb{R} \) are *separated* if \( \overline{A} \cap B = \emptyset \) and \( A \cap \overline{B} = \emptyset \).

A set \( E \subseteq \mathbb{R} \) is *disconnected* if it can be written as \( E = A \cup B \) for separated sets \( A \) and \( B \).

A set that is not disconnected is called a **connected** set.

**Example 3.4.5.** (i) The set \( A = (1, 2) \) and \( B = (2, 5) \) are separated because

\[
\overline{A} \cap B = [1, 2] \cap (2, 5) = \emptyset, \quad A \cap \overline{B} = (1, 2) \cap [2, 5] = \emptyset.
\]

The set \( E = A \cup B \) is disconnected because it is the union of the separated sets \( A \) and \( B \).

On the other hand, the sets \( C = (1, 2] \) and \( D = (2, 5) \) are not separated because \( C \cap \overline{D} = \{2\} \).

The set \( C \cup D \) is the interval \((1, 5)\) which is connected (although we have not shown this).

(ii) The set of rational numbers is disconnected.

To see this we set

\[
A = \mathbb{Q} \cap (-\infty, \sqrt{2}), \quad B = \mathbb{Q} \cap (\sqrt{2}, \infty).
\]

We certainly have \( \mathbb{Q} = A \cup B \).

The Order Limit Theorem implies that any limit point of \( A \) will be in \((-\infty, \sqrt{2}]\), which is disjoint from \( B \).

Similarly, \( A \cap \overline{B} \neq \emptyset \), and so \( A \) and \( B \) are separated sets.

We conclude that \( \mathbb{Q} \) is disconnected.

By carefully working through the logical negations of the quantifiers in the definition of disconnected, we arrive at a positive characterization of connectedness.

**Theorem 3.4.6.** A set \( E \subseteq \mathbb{R} \) is connected if and only if, for all nonempty disjoint sets \( A \) and \( B \) satisfying \( E = A \cup B \), there always exists a convergent sequence \( (x_n) \) with all \( x_n \) contained in one of \( A \) or \( B \), and \( x = \lim x_n \) contained in the other.

This notion of connectedness is more relevant in higher dimensions, for in dimension 1, a subset \( E \subseteq \mathbb{R} \) is connected precisely when \( E \) is an interval.

**Theorem 3.4.7.** A set \( E \subseteq \mathbb{R} \) is connected if and only if whenever \( a < c < b \) with \( a, b \in E \), it follows for that \( c \in E \) too.

**Proof.** Suppose that \( E \) is connected, let \( a, b \in E \), and pick \( a < c < b \).

Suppose \( c \not\in E \), and set

\[
A = (-\infty, c) \cap E, \quad B = (c, \infty) \cap E.
\]

Because \( a \in A \) and \( b \in B \), both \( A \) and \( B \) are nonempty; and \( E = A \cup B \).
Since any limit point of $l$ of $A$ satisfies $l \leq c$ by the Order Limit Theorem, we have that $\overline{A} \cap B = \emptyset$.

Similarly, we have $A \cap \overline{B} = \emptyset$.

Thus $A$ and $B$ are separated set, and so $E = A \cup B$ is disconnected, a contradiction.

Hence, $c \in E$.

Now suppose whenever $a < c < b$ with $a, b \in E$ we have that $c \in E$ too.

We will use Theorem 3.4.6 to show that $E$ is connected.

To this end we write $E = A \cup B$ for nonempty disjoint sets $A$ and $B$.

Pick $a_0 \in A$ and $b_0 \in B$, and WLOG suppose that $a_0 < b_0$.

Since every $c \in (a_0, b_0)$ must be in $E$, we have that $I_0 = [a_0, b_0] \subseteq E$.

Bisect $I_0$ into two equal halves.

The midpoint of $I_0$ is either in $A$ or $B$.

If the midpoint of $I_0$ is in $A$, take $I_1 = [a_1, b_1]$ to be the right half where $a_1$ is the midpoint of $I_0$ and $b_1 = b_0 \in B$.

If the midpoint of $I_0$ is in $B$, take $I_1 = [a_1, b_1]$ to be the left half where $b_1$ is the midpoint of $I_0$ and $a_1 = a_0 \in A$.

Continuing this process yields a sequence of nested intervals $I_n = [a_n, b_n]$ where $a_n \in A$ and $b_n \in B$, and whose lengths $b_n - a_n$ go to 0 as $n \to \infty$.

By the Nested Interval Property, there exists

$$x \in \bigcap_{n=0}^{\infty} I_n.$$ 

The sequence $(a_n)$ of left endpoints belongs to $A$ and converges to $x$, and the sequence $(b_n)$ of right endpoints belongs to $B$ and converges to $x$ as well.

[Note: $(a_n)$ and $(b_n)$ are equivalent Cauchy sequences.]

Since $x \in I_0$ and $I_0 \subseteq E$, we have that $x \in E = A \cup B$, which means that $x \in A$ or $x \in B$.

So there is a limit point of one of $A$ or $B$ that belongs to the other, and by Theorem 3.4.6, the set $E$ is connected.