Math 341 Lecture #19
§4.1: Examples of Dirichlet and Thomae

We begin a discussion about the continuity of a function \( f: A \to \mathbb{R} \), for a nonempty \( A \subset \mathbb{R} \).

Recall from Calculus I that we say \( f \) is continuous at a point \( a \in \mathbb{R} \) if \( f(a) \) exists (i.e., \( a \in A \)), \( \lim_{x \to a} f(x) \) exists, and \( \lim_{x \to a} f(x) = f(a) \).

[We are leaving the notion of the limit of a function vague for now; we will see the rigorous definition next time.]

For a function \( f: A \to \mathbb{R} \), we let \( D_f \) denote the set of points in \( A \) where \( f \) is not continuous.

What kind of a subset of \( \mathbb{R} \) can \( D_f \) be?

**Example.** Dirichlet defined a function \( g: \mathbb{R} \to \mathbb{R} \) by

\[
g(x) = \begin{cases} 
1 & \text{if } x \in \mathbb{Q}, \\
0 & \text{if } x \notin \mathbb{Q}.
\end{cases}
\]

For any \( c \in \mathbb{R} \) we can find sequences \( (x_n) \) in \( \mathbb{Q} \) and \( (y_n) \) in \( \mathbb{Q}^c \) such that \( x_n \to c \) and \( y_n \to c \), but for which \( g(x_n) = 1 \) and \( g(y_n) = 0 \) for all \( n \in \mathbb{N} \), so that

\[
\lim_{n \to \infty} g(x_n) \neq \lim_{n \to \infty} g(y_n).
\]

This suggests that \( f \) is not continuous at \( c \), and as \( c \) was arbitrary, that \( f \) is not continuous at any \( c \in \mathbb{R} \).

We have that \( D_g = \mathbb{R} \).

**Example.** A modification of Dirichlet’s function results in a function that is continuous at just one point.

Define \( h: \mathbb{R} \to \mathbb{R} \) by

\[
h(x) = \begin{cases} 
x & \text{if } x \in \mathbb{Q}, \\
0 & \text{if } x \notin \mathbb{Q}.
\end{cases}
\]

For a nonzero \( c \) we can find sequences \( (x_n) \) in \( \mathbb{Q} \) and \( (y_n) \) in \( \mathbb{Q}^c \) such that \( x_n \to c \) and \( y_n \to c \), but for which \( h(x_n) = x_n \) and \( h(y_n) = 0 \) for all \( n \in \mathbb{N} \), so that

\[
\lim_{n \to \infty} h(x_n) = c \neq 0 = \lim_{n \to \infty} h(y_n).
\]

This suggests that the function \( h \) is not continuous at any point \( c \neq 0 \).

However, if \( c = 0 \), then for any sequence \( (z_n) \) in \( \mathbb{R} \) with \( z_n \to 0 \) we have \( |h(z_n)| \leq |z_n| \), so that \( h(z_n) \to 0 \) as well.

Thus \( h \) is continuous at \( c = 0 \).

We have that \( D_h = \mathbb{R} - \{0\} \).
Example. Thomae defined a function \( t : \mathbb{R} \to \mathbb{R} \) by

\[
t(x) = \begin{cases} 
1 & \text{if } x = 0, \\
1/n & \text{if } x = m/n \in \mathbb{Q} \setminus \{0\} \text{ in lowest terms with } n > 0, \\
0 & \text{if } x \notin \mathbb{Q}.
\end{cases}
\]

For \( c \in \mathbb{Q} \), we have \( t(c) > 0 \).
For a sequence \( (y_n) \in \mathbb{Q}^c \) such that \( y_n \to c \), we have \( t(y_n) = 0 \) for all \( n \in \mathbb{N} \), so that

\[
t(c) \neq 0 = \lim_{n \to \infty} t(y_n).
\]

This suggests that \( t \) is discontinuous at every rational point.
On the other hand, if \( c \) is irrational, we have \( t(c) = 0 \).
For any sequence \( (x_n) \) in \( \mathbb{R} \) such that \( x_n \to c \) we have \( t(x_n) = 0 \) when \( x_n \notin \mathbb{Q} \) or \( t(x_n) \) is the reciprocal of the positive denominator of the rational \( x_n \) is lowest terms.
The closer \( x_n \) is to the irrational \( c \), the larger the denominator of \( x_n \) is, so that \( t(x_n) \) is as close to 0 as needed.
The result of this is that \( t(x_n) \to 0 \) as \( n \to \infty \), that is, we have

\[
\lim_{n \to \infty} t(x_n) = 0 = t(c),
\]
suggesting that \( t \) is continuous at every irrational \( c \).
We have that \( D_t = \mathbb{Q} \).

Example. Define a function \( s : \mathbb{R} \to \mathbb{R} \) by

\[
s(x) = [\lfloor x \rfloor]
\]
where \([x]\) is the largest integer \( n \) such that \( n \leq s \).
For \( c \in \mathbb{R} \) such that \( n < c < n + 1 \) for \( n \in \mathbb{N} \), we have for any sequence \( (x_n) \) converging to \( c \) that

\[
\lim_{n \to \infty} s(x_n) = n = [\lfloor c \rfloor].
\]

On the other hand, for \( c = n \) for \( n \in \mathbb{N} \), we take a sequence \( (y_n) \) such that \( n - 1 < y_n < n \) and \( y_n \to c \), so that

\[
\lim_{n \to \infty} s(y_n) = n - 1 \neq [\lfloor c \rfloor] = n.
\]
This suggests that \( s \) is discontinuous at every integer point, and we have that \( D_s = \mathbb{Z} \).

Example. Define a function \( f : \mathbb{R} \to \mathbb{R} \) by

\[
f(x) = \begin{cases} 
0 & \text{if } x \leq 0, \\
x & \text{if } x \in \mathbb{Q} \cap (0, 1), \\
0 & \text{if } x \in (0, 1) - \mathbb{Q}, \\
0 & \text{if } x \geq 1.
\end{cases}
\]
The function is continuous at every \( c < 0 \) and at every \( c > 1 \).

As with the modified Dirichlet function, this function \( f \) is continuous at \( c = 0 \), but discontinuous at every \( c \in (0, 1) \).

This function is also discontinuous at \( c = 1 \) because for a rational sequence \((x_n)\) in \((0, 1)\) with \( x_n \to 1 \) we have \( f(x_n) = x_n \to 1 \), while for any sequence \((y_n)\) with \( y_n > 1 \) and \( y_n \to 1 \) we have \( f(y_n) \to 0 \).

So here we have \( D_f = (0, 1] \).

With all of the examples we have explored, what is the topological property shared by the set of discontinuities? Open, closed, compact, connected, \( F_\sigma, G_\delta \)?

If you are thinking an \( F_\sigma \) set, you are correct.

To prove this is somewhat involved, so we focus in Section 4.6 on a simpler class of functions \( f \) for which \( D_f \) is more readily understood.