Math 341 Lecture #21
§4.3: Continuous Functions

We recall the \( \epsilon - \delta \) notion of what it means for a function to be continuous at a point in its domain.

**Definition 4.3.1.** For a nonempty \( A \subseteq \mathbb{R} \), a function \( f : A \rightarrow \mathbb{R} \) is continuous at a point \( c \in A \) if, for all \( \epsilon > 0 \), there exists \( \delta > 0 \) such that whenever \( |x - c| < \delta \) and \( x \in A \), it follows that \( |f(x) - f(c)| < \epsilon \).

If \( f \) is continuous at every point of \( A \), we say that \( f \) is continuous on \( A \).

This definition of continuity at a point looks like a functional limit, except that for continuity we require that \( c \in A \), not merely that \( c \) is a limit point of \( A \).

We would like to say \( f \) is continuous at \( c \in A \) by writing

\[
\lim_{x \to c} f(x) = f(c).
\]

The minor technical difficulty with doing this is when \( c \in A \) is an isolated point, but then \( f \) is continuous at \( c \) in this case because \( V_\delta(c) \cap A = \{c\} \) for all small enough \( \delta > 0 \).

**Theorem 4.3.2 (Characterization of Continuity).** For a nonempty \( A \subseteq \mathbb{R} \), let \( f : A \rightarrow \mathbb{R} \), and let \( c \in A \). Then \( f \) is continuous at \( c \) if and only if any one of the following conditions is met:

(i) for all \( \epsilon > 0 \) there exists \( \delta > 0 \) such that \( |x - c| < \delta \) and \( x \in A \) implies \( |f(x) - f(c)| < \epsilon \);

(ii) for all \( V_\epsilon(f(c)) \) there is \( V_\delta(c) \) for which \( x \in V_\delta(c) \cap A \) implies \( f(x) \in V_\epsilon(f(c)) \);

(iii) if \( (x_n) \to c \) with \( x_n \in A \), then \( f(x_n) \to f(c) \).

If \( c \) is a limit point of \( A \), then the above conditions are equivalent to

(iv) \( \lim_{x \to c} f(x) = f(c) \);

Proof. The equivalence of (i), (iii), and (iv) is simple and straightforward.

We will show the equivalence of (ii) and (iii).

Suppose for all \( V_\epsilon(f(c)) \) there is \( V_\delta(c) \) for which \( x \in V_\delta(c) \cap A \) implies \( f(x) \in V_\epsilon(f(c)) \).

Let \( (x_n) \) be a sequence in \( A \) converging to \( c \).

Thus there exists \( N \in \mathbb{N} \) such that \( x_n \in V_\delta(x) \cap A \) for all \( n \geq N \).

This implies that \( f(x_n) \in V_\epsilon(f(c)) \) and so we have \( f(x_n) \to f(c) \).

Now, suppose, to the contrary, that there exists \( \epsilon_0 \) such that for all \( V_\delta(c) \) there exists \( x \in V_\delta(c) \cap A \) such that \( f(x) \not\in V_{\epsilon_0}(f(c)) \).

For each \( \delta_n = 1/n \) we choose \( x_n \in V_{\delta_n}(c) \cap A \) for which \( f(x_n) \not\in V_{\epsilon_0}(f(c)) \).

This gives a sequence \( (x_n) \) in \( A \) which converges to \( c \), but for which \( f(x_n) \not\to f(c) \). \( \square \)

Statement (iii) of this theorem is a new approach to continuity, and hence to discontinuity.
Corollary 4.3.3 (Criterion for Discontinuity). Let \( f : A \to \mathbb{R} \), and let \( c \in A \) be a limit point of \( A \). If there exists a sequence \((x_n)\) in \( A \) with \( x_n \to c \) such that \( f(x_n) \not\to f(c) \), then \( f \) is not continuous at \( c \).

Using the sequential approach to continuity establishes the following.

Theorem 4.3.4 (Algebraic Continuity Theorem). Assume \( f : A \to \mathbb{R} \) and \( g : A \to \mathbb{R} \) are continuous at \( c \in A \). Then

(i) \( kf(x) \) is continuous at \( c \) for all \( k \in \mathbb{R} \),
(ii) \( f(x) + g(x) \) is continuous at \( c \),
(iii) \( f(x)g(x) \) is continuous at \( c \), and
(iv) \( f(x)/g(x) \) is continuous at \( c \) provided the quotient is defined.

Example 4.3.5. For constants \( a, b \in \mathbb{R} \), the function \( f(x) = ax + b \) for \( x \in A = \mathbb{R} \) is continuous.

To see why, take \( c \in A \), and consider

\[
|f(x) - (ac + b)| = |ax + b - ac - b| = |a| |x - c|.
\]

For \( \epsilon > 0 \) we choose \( \delta = \epsilon/|a| \), so that when \( x \in V_\delta(c) \) we have

\[
|f(x) - (ac + b)| < |a|\epsilon/|a| = \epsilon.
\]

By Theorem 4.3.4, it follows that every polynomial is continuous on \( \mathbb{R} \), and that every rational function is continuous at those points where the denominator is not zero.

Example 4.3.6. Is the function

\[
g(x) = \begin{cases} 
    x \sin(1/x) & \text{if } x \neq 0, \\
    0 & \text{if } x = 0,
\end{cases}
\]

continuous at \( c = 0 \)?

For \( x \neq 0 \), we have

\[
|g(x) - g(0)| = |x \sin(1/x) - 0| = |x \sin(1/x)| \leq |x|
\]

because \( |\sin(1/x)| \leq 1 \).

For \( \epsilon > 0 \) we choose \( \delta = \epsilon \) so that when \( x \in V_\delta(0) \) we have

\[
|g(x) - g(0)| \leq |x| < \delta = \epsilon.
\]

Thus \( g(x) \) is continuous at \( c = 0 \).

Example 4.3.8. Show that \( f(x) = \sqrt{x} \) is continuous on \( A = [0, \infty) \).

For \( \epsilon > 0 \) and \( c \in A \), we need to find \( \delta > 0 \) such that

\[
|f(x) - f(c)| = |\sqrt{x} - \sqrt{c}| < \epsilon
\]
when $x \in V_\delta(c) \cap A$.

When $c = 0$ this reduces to $|\sqrt{x} - 0| = \sqrt{x} < \epsilon$.

This gives $x < \epsilon^2$ from which we choose $\delta = \epsilon^2$.

Thus for $x \in V_\delta(0) \cap A$ we have $|\sqrt{x} - 0| = \sqrt{x} < \sqrt{\delta} = \sqrt{\epsilon^2} = \epsilon$, and so we have continuity of $g(x)$ at $c = 0$.

Now for $c > 0$ we are dealing with

$$|\sqrt{x} - \sqrt{c}| = |\sqrt{x} - \sqrt{c}| \frac{\sqrt{x} + \sqrt{c}}{\sqrt{x} + \sqrt{c}} = \frac{|x - c|}{\sqrt{x} + \sqrt{c}}.$$  

We can control the numerator $|x - c|$ by the choice of $\delta$, so we need to find a way to deal with the denominator $\sqrt{x} + \sqrt{c}$.

Since $\sqrt{x} + \sqrt{c} \geq \sqrt{c} > 0$ we have

$$\frac{1}{\sqrt{x} + \sqrt{c}} \leq \frac{1}{\sqrt{c}}.$$  

We then have

$$|\sqrt{x} - \sqrt{c}| \leq \frac{|x - c|}{\sqrt{c}}.$$  

The choice of $\delta = \epsilon \sqrt{c}$ then gives us

$$|\sqrt{x} - \sqrt{c}| < \frac{\epsilon \sqrt{c}}{\sqrt{c}} = \epsilon.$$  

Thus $f(x) = \sqrt{x}$ is continuous at $c > 0$, and so $f(x)$ is continuous on $A$.

Recall that the range of a function $f : A \to \mathbb{R}$ is the set $f(A) = \{f(x) : x \in A\}$.

Theorem 4.3.9 (Composition of Continuous Functions). Suppose for functions $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ satisfy $f(A) \subseteq B$. If $f$ is continuous at $c \in A$ and $g$ is continuous at $f(c)$, then the composition $g \circ f(x)$ is continuous at $c$.

The proof of this is a homework problem (4.3.3).