Math 341 Lecture #24
§4.6: Sets of Discontinuity

We saw at the beginning of Chapter 4 that the set of discontinuities \( D_f \) for a function \( f : A \to \mathbb{R} \) appeared to always be an \( F_\sigma \) set (a countable union of closed sets).

We will prove this in the case when \( f \) is monotone.

**Definition.** 4.6.1. A function \( f : A \to \mathbb{R} \) is *increasing* on \( A \) if \( f(x) \leq f(y) \) whenever \( x < y \) for \( x, y \in A \), and is *decreasing* if \( f(x) \geq f(y) \) whenever \( x < y \) for \( x, y \in A \).

A function \( f : A \to \mathbb{R} \) is *monotone* if \( f \) is either increasing or decreasing.

The function \( s(x) = \lfloor x \rfloor \) on \( \mathbb{R} \) is monotone increasing.

In showing that \( s(x) \) is discontinuous at every integer point, we took a sequence \( y_n \) such that \( n - 1 < y_n < n \) and \( y_n \to n \).

This is a sequence that approaches \( n \) from the left.

We can talk about functional limits in the same way: from the left or from the right.

**Definition.** 4.6.2. Given a limit point \( c \) of a nonempty set \( A \) and a function \( f : A \to \mathbb{R} \) we say the limit of \( f(x) \) exists from the right and equals \( L \), and write

\[
\lim_{x \to c^+} f(x) = L,
\]

if for all \( \epsilon > 0 \) there exists \( \delta > 0 \) such that \( |f(x) - L| < \epsilon \) whenever \( 0 < x - c < \delta \) and \( x \in A \).

In terms of sequences this is the same as \( (x_n) \) in \( A \) with \( x_n > c \) and \( x_n \to c \), for which \( f(x_n) \to L \).

You have it as a homework problem (4.6.3) to state the definition of the limit from the left,

\[
\lim_{x \to c^-} f(x) = L.
\]

Recall that the limits from the right and from the left are related to the limit.

**Theorem 4.6.3.** Let \( f : A \to \mathbb{R} \) and \( c \) a limit point of \( A \). Then \( \lim_{x \to c} f(x) = L \) if and only if

\[
\lim_{x \to c^-} f(x) = L \quad \text{and} \quad \lim_{x \to c^+} f(x) = L.
\]

The discontinuities of a function can be divided into three categories.

(i) If \( \lim_{x \to c^-} f(x) \) exists but is not equal to \( f(c) \), then \( f \) has a *removable* discontinuity at \( c \).

(ii) If \( \lim_{x \to c^-} f(x) \) and \( \lim_{x \to c^+} f(x) \) both exist but are not equal, then \( f \) has a *jump* discontinuity at \( c \).

(iii) If \( \lim_{x \to c^-} f(x) \) does not exist for some other reason, then \( f \) has an *essential* discontinuity at \( c \).
The third category includes vertical asymptote type discontinuities, like $f(x) = \frac{1}{x}$ has at $x = 0$, and bounded oscillatory type discontinuities, like $f(x) = \sin(\frac{1}{x})$ has at $x = 0$.

A monotone function $f$, though, can have only one type of discontinuity, and this is what makes it easier to identify $D_f$ in this case.

**Theorem.** If $f : \mathbb{R} \to \mathbb{R}$ is monotone, then

$$\lim_{x \to c^-} f(x) \text{ and } \lim_{x \to c^+} f(x)$$

exist at every at point $c$ in $\mathbb{R}$.

**Proof.** WLOG, suppose that $f$ is increasing.

For $c \in \mathbb{R}$ consider the nonempty subset $B = \{y = f(x) : x < c\}$ of $\mathbb{R}$.

Since $f$ is increasing, the number $f(c)$ is an upper bound for $A$.

By the Axiom of Completeness, the number $\sup B$ exists.

The claim is that

$$\lim_{x \to c^-} f(x) = \sup B.$$

For $L = \sup B$, we have that for all $\epsilon > 0$ there exist $y_\epsilon \in B$ such that $L - \epsilon < y_\epsilon \leq L$.

Since $y_\epsilon \in B$, there is $x_\epsilon < c$ such that $f(x_\epsilon) = y_\epsilon$.

For any sequence $(x_n)$ with $x_n < c$ and $x_n \to c$, there exists $N \in \mathbb{N}$ such that $x_\epsilon \leq x_n < c$ for all $n \geq N$.

Thus using the monotonicity of $f$, we have

$$L - \epsilon < y_\epsilon = f(x_\epsilon) \leq f(x_n) \leq L < L + \epsilon \text{ for all } n \geq N.$$

This says that $f(x_n) \to L$, and so $\lim_{x \to c^-} f(x)$ exists.

In a similar manner we show that $\lim_{x \to c^+} f(x)$ exists. \qed

**Corollary (Exercise 4.6.5).** A monotone function $f : \mathbb{R} \to \mathbb{R}$ can have only jump discontinuities.

**Proof.** By the Theorem, we have for each $c \in \mathbb{R}$ that

$$\lim_{x \to c^-} f(x), \lim_{x \to c^+} f(x)$$

both exist.

When these two limits agree, the function $f$ is continuous at $c$ by Theorem 4.6.3.

When these two limits disagree, the function $f$ has a jump discontinuity with a jump of

$$\lim_{x \to c^+} f(x) - \lim_{x \to c^-} f(x)$$

at $c$.

The only discontinuities that a monotone function can have are jump discontinuities. \qed
Recall that the monotone function \( s(x) = \lfloor x \rfloor \) on \( \mathbb{R} \) has \( D_s = \mathbb{Z} \), i.e., a countable set of points where \( s(x) \) is not continuous.

You have it as a homework problem (4.6.6) to show for a monotone function \( f \) that there exists a bijection between \( D_f \) and a subset of \( \mathbb{Q} \).

Since every subset of \( \mathbb{Q} \) is an \( F_\sigma \) set, we will have shown that \( D_f \) is an \( F_\sigma \) set when \( f \) is monotone.