Math 341 Lecture #25
§5.2: Derivatives

We now begin a rigorous exploration of the notion of differentiability for a function.

Definition 5.2.1. Let $g : A \to \mathbb{R}$ be a function where A is an interval. For $c \in A$, the *derivative* of g at c is defined by

$$g'(c) = \lim_{x \to c} \frac{g(x) - g(c)}{x - c}$$

provided the limit exists.

If $g'(c)$ exists for all $c \in A$, we say that g is *differentiable on* A.

Note that in this definition of differentiability the domain A is an *interval*, i.e, a connected subset of $\mathbb{R}$, and hence every $c \in A$ is a limit point of A.

The definition of derivative allows for one-sided derivatives when $c \in A$ is an endpoint of A.

Theorem 5.2.3. If $g : A \to \mathbb{R}$ is differentiable at $c \in A$, then g is continuous at c as well.

Proof. Suppose that g is differentiable at c, i.e.,

$$g'(c) = \lim_{x \to c} \frac{g(x) - g(c)}{x - c}$$

exists.

By the Algebraic Limit Theorem we have that

$$\lim_{x \to c} \big(g(x) - g(c)\big) = \lim_{x \to c} \left(\frac{g(x) - g(c)}{x - c}\right) \big(x - c\big) = g'(c) \cdot 0 = 0.$$

This implies that

$$\lim_{x \to c} g(x) = g(c),$$

and so g is continuous at c. $\square$

Algebraic combinations of functions differentiable at a point yields functions differentiable at that point.

Theorem 5.2.4. Let f and g be functions defined on an interval A, and suppose that f and g are both differentiable at $c \in A$. Then

(i) $(f + g)'(c) = f'(c) + g'(c)$,
(ii) $(kf)'(c) = kf'(c)$ for any constant $k \in \mathbb{R}$,
(iii) $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$, and
(iv) $(f/g)'(c) = \dfrac{f'(c)g(c) - f(c)g'(c)}{[g(c)]^2}$ provided $g(c) \neq 0$.

Proof. Statements (i), (ii), and (iv) are left for you.

For (iii) we have

$$\begin{aligned}\frac{(fg)(x)-(fg)(c)}{x-c} &= \frac{f(x)g(x)-f(x)g(c)+f(x)g(c)-f(c)g(c)}{x-c} \\ &= f(x)\left[\frac{g(x)-g(c)}{x-c}\right]+g(c)\left[\frac{f(x)-f(c)}{x-c}\right].\end{aligned}$$

Because f is differentiable at c, by Theorem 5.2.3, we have that $f(x)\to f(c)$ as $x\to c$.

Because f and g are differentiable at c, the limits of the terms in the square brackets as $x\to c$ exist and equal $g'(c)$ and $f'(c)$ respectively, thus giving (iii). □

The composition of two differentiable functions is also differentiable.

Theorem 5.2.5 (Chain Rule). Let $f: A\to\mathbb{R}$ and $g: B\to\mathbb{R}$ such that $f(A)\subseteq B$. If f is differentiable at $c\in A$ and g is differentiable at $f(c)$ in B, then $g\circ f$ is differentiable at c with

$$(g\circ f)'(c) = g'(f(c))f'(c).$$

Proof. With g differentiable at $f(c)$ we have

$$g'(f(c)) = \lim_{y\to f(c)}\frac{g(y)-g(f(c))}{y-f(c)}.$$

Another way to assert this limit is by saying the limit of

$$d(y) = \frac{g(y)-g(f(c))}{y-f(c)} - g'(f(c))$$

is 0 as $y\to f(c)$.

The function $d(y)$ is not defined when $y=f(c)$, but we declare $d(f(c))=0$, so that d is continuous at $f(c)$.

We rewrite the equation defining $d(y)$ as

$$g(y)-g(f(c)) = [g'(f(c))+d(y)](y-f(c)).$$

Notice that this equation holds for all $y\in B$ including when $y=f(c)$.

We can then substitute $f(t)$ for y for any $t\in A$ that we wish, to give

$$g(f(t))-g(f(c)) = [g'(f(c))+d(f(t))](f(t)-f(c)).$$

For $t\neq c$, we divide both sides of this by $t-c$ to get

$$\frac{g(f(t))-g(f(c))}{t-c} = [g'(f(t))+d(f(t))]\frac{f(t)-f(c)}{t-c}.$$

For the right-hand side of this, we have that $d(f(t))\to 0$ because $f(t)\to f(c)$ by the continuity of f at c implied by the differentiability of f at c.

Thus by the differentiability of f at c, the limit of the right-hand side exists and is equal to $g'(f(c))f'(c)$.

This implies the limit of the left-hand side exists, and so $g \circ f$ is differentiable at c. $\square$

What can we say about the derivative of a differentiable function, especial when the derivative is not continuous?

Theorem 5.2.6 (Interior Extremum Theorem). Let f be differentiable on an open interval (a, b). If f attains a maximum value at some point $c \in (a, b)$, then $f'(c) = 0$. The same is true if $f(c)$ is a minimum value.

Proof. Because $c \in (a, b)$ we can find sequences (x_n) and (y_n) in (a, b) for which $x_n < c < y_n$ for all $n \in \mathbb{N}$ and $x_n \to c$ and $y_n \to c$.

Since $f(c)$ is a maximum value, we have that $f(y_n) - f(c) \leq 0$ for all $n \in \mathbb{N}$, and thus with $y_n - c > 0$ we have

$$f'(c) = \lim_{n\to\infty} \frac{f(y_n) - f(c)}{y_n - c} \leq 0$$

by the Order Limit Theorem.

Similary we have $f(x_n) - f(c) \leq 0$ and $x_n - c < 0$ so that

$$f'(c) = \lim_{n\to\infty} \frac{f(x_n) - f(c)}{x_n - c} \geq 0.$$

These implies that $f'(c) = 0$. $\square$

All of the functions for which you found the maximum or minimum values in Calculus had continuous derivatives.

In this case the Intermediate Value Theorem applies: for f' continuous on $[a, b]$ means that for every value L between $f'(a)$ and $f'(b)$ there is $c \in (a, b)$ such that $f'(c) = L$.

Does this happens when the derivative of a differentiable function is not continuous?

Theorem 5.2.7 (Darboux's Theorem). If f is differentiable on an interval $[a, b]$, then f' has the intermediate value property on $[a, b]$.

Start of Proof. For $s, t \in [a, b]$ with $s < t$ suppose there is α between $f'(s)$ and $f'(t)$.

We are looking for $c \in (s, t)$ such that $f'(c) = \alpha$.

We convert this problem to one of finding a root c of the derivative of the differentiable function $g(x) = f(x) - \alpha x$.

Here $g'(x) = f'(x) - \alpha$, and we are looking for $c \in (s, t)$ such that $g'(c) = f'(c) - \alpha = 0$.

WLOG suppose that $f'(s) < \alpha < f'(t)$.

Then $g'(s) = f'(s) - \alpha < 0$ and $g'(t) = f'(t) - \alpha > 0$.

The remainder of the proof is a homework problem (5.2.11).