The simple “observation” for a differentiable function \( f : [a, b] \rightarrow \mathbb{R} \) that there exists \( c \in (a, b) \) such that

\[
f'(c) = \frac{f(b) - f(a)}{b - a}
\]

is known as the Mean Value Theorem, and is the cornerstone of almost every major theorem about differentiation there is (as we shall see).

**Theorem 5.3.1 (Rolle’s Theorem).** Let \( f : [a, b] \rightarrow \mathbb{R} \) be continuous on \([a, b]\), differentiable on \((a, b)\). If \( f(a) = f(b) \), then there exists \( c \in (a, b) \) such that \( f'(c) = 0 \).

**Proof.** The continuity of \( f \) on the compact set \([a, b]\) implies that \( f \) attains a maximum value and a minimum value.

If the maximum value occurs at one endpoint of \([a, b]\), and the minimum value occurs at the other endpoint of \([a, b]\), then \( f \) is a constant function, so that \( f'(x) = 0 \) for all \( x \in (a, b) \), and we can choose any \( c \in (a, b) \).

If the maximum value or the minimum value occurs at an interior point \( c \) of \([a, b]\), then by the Interior Extremum Theorem we have \( f'(c) = 0 \).

**Theorem 5.3.2 (Mean Value Theorem).** If \( f : [a, b] \rightarrow \mathbb{R} \) is continuous on \([a, b]\), differentiable on \((a, b)\), then there exists \( c \in (a, b) \) such that

\[
f'(c) = \frac{f(b) - f(a)}{b - a}.
\]

**Proof.** We reduce the general case of the Mean Value Theorem to the special case of Rolle’s Theorem by constructing a new function from \( f \).

Define a function \( g : [a, b] \rightarrow \mathbb{R} \) by

\[
g(x) = f(x) - \left( \frac{f(b) - f(a)}{b - a} \right) (x - a) + f(a).
\]

This function \( g(x) \) measures the vertical distance from the graph of \( f(x) \) to the graph of the line connecting the points \((a, f(a))\) and \((b, f(b))\).

One readily verifies that \( g \) is continuous on \([a, b]\) (because \( f \) is), and that \( g \) is differentiable on \((a, b)\) (because \( f \) is).

We evaluate \( g \) at the endpoints:

\[g(a) = f(a) - [0 + f(a)] = 0, \quad g(b) = f(b) - [f(b) - f(a) + f(a)] = 0.\]

We can now applies Rolle’s Theorem to obtain the existence of \( c \in (a, b) \) such that \( g'(c) = 0 \).

We then translate this back to \( f \):

\[0 = g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}.\]
Rearranging this gives

\[ f'(c) = \frac{f(b) - f(a)}{b - a} \]

which is the desired result. □

We know for a constant function \( f(x) = k \) on an interval \( A \) that \( f'(x) = 0 \) for all \( x \in A \).

But how do we prove the converse? With the Mean Value Theorem.

**Corollary 5.3.3.** For an interval \( A \), if \( g : A \to \mathbb{R} \) is differentiable and satisfies \( g'(x) = 0 \) for all \( x \in A \), then \( g(x) = k \) for some constant \( k \in \mathbb{R} \).

**Proof.** Take \( x, y \in A \) and WLOG suppose \( x < y \).

Applying the Mean Value Theorem to \( g \) on \([x, y]\) there exists \( c \in (x, y) \) such that

\[ g'(c) = \frac{g(y) - g(x)}{y - x}. \]

By hypothesis, we have that \( g'(c) = 0 \), so that as \( x \neq y \), we have \( g(x) = g(y) \).

Set \( k \) equal to this common value.

The arbitrariness of \( x \) and \( y \) now implies that \( g(x) = k \) for all \( x \in A \). □.

Another consequence of the Mean Value Theorem is the familiar result that two antiderivatives of a continuous function differ by a constant.

**Corollary 5.3.4.** If \( f \) and \( g \) are differentiable on an interval \( A \) and satisfy \( f'(x) = g'(x) \) for all \( x \in A \), then \( f(x) = g(x) + k \) for some constant \( k \in \mathbb{R} \).

**Proof.** The function \( h(x) = f(x) - g(x) \) is differentiable on \( A \) and satisfies \( h'(x) = 0 \) for all \( x \in A \).

By Corollary 5.3.3, we know that \( h(x) = k \) for some constant \( k \in \mathbb{R} \), so that \( f(x) = g(x) + k \). □

**Example.** Let \( f \) be a differentiable function on \([0, 3]\) where \( f(0) = 1, f(1) = 3, f(2) = 1, \) and \( f(3) = 2, \) and \( f'(x) \geq 1 \) for \( x \in (0, 1) \).

Now there will be lots of differentiable functions whose graphs pass through the points \((0, 1), (1, 3), (2, 1), \) and \((3, 2), \) and have \( f'(x) \geq 1 \) for all \( x \in (0, 1) \).

What properties do all of these differentiable functions have?

Since \( f'(x) \geq 1 \) for all \( x \in (0, 1) \), we know that \( f \) is increasing on \((0, 1) \).

The function \( g(x) = x - f(x) \) is continuous on \([0, 3]\), and since \( g(0) = -1 \) and \( g(3) = 3 - 2 = 1 \), there exists by the Intermediate Value Theorem a point \( d \in (0, 3) \) such that \( g(d) = 0, \) or \( f(d) = d \).

That is, the graph of \( f \) crosses the graph of \( y = x \) at \( x = d \).

For any subinterval \([a, b]\) of \([0, 3]\), the function \( f \) is continuous on \([a, b]\) and differentiable on \((a, b)\).
By the Mean Value Theorem there exists $c_1 \in (1, 2)$ such that

$$f'(c_1) = \frac{f(2) - f(1)}{2 - 1} = \frac{1 - 3}{1} = -2.$$ 

By the Mean Value Theorem there exists $c_2 \in (2, 3)$ such that

$$f'(c_2) = \frac{f(3) - f(2)}{3 - 2} = \frac{2 - 1}{3 - 2} = 1.$$ 

Since $f$ is differentiable on $[c_1, c_2]$, there is by Darboux’s Theorem a point $c_3 \in (c_1, c_2) \subseteq (1, 3)$ such that $f'(c_3) = 0$.

The following result is a generalization of the Mean Value Theorem due to Cauchy, and is key to proving L’Hospital’s Rule.

**Theorem 5.3.5.** If $f$ and $g$ are continuous on $[a, b]$ and differentiable on $(a, b)$, then there exists $c \in (a, b)$ such that

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c).$$

If $g'(x) \neq 0$ for all $x \in (a, b)$, then we have

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$ 

Proof. The function

$$h(x) = [f(b) - f(a)]g(x) - [g(b) - g(a)]f(x)$$

is continuous on $[a, b]$ and differentiable on $(a, b)$.

Since

$$h(b) = [f(b) - f(a)]g(b) - [g(b) - g(a)]f(b)$$

$$= f(b)g(b) - f(a)g(b) - g(b)f(b) + g(a)g(b)$$

$$= -f(a)g(b) + g(a)f(b),$$

$$h(a) = [f(b) - f(a)]g(a) - [g(b) - g(a)]f(a)$$

$$= f(b)g(a) - f(a)g(a) - g(b)f(a) + g(a)f(a)$$

$$= f(b)g(a) - g(b)f(a),$$

we have that $h(b) = h(a)$.

Applying Rolle’s Theorem gives the existence of $c \in (a, b)$ such that $h'(c) = 0$, which unwrapped gives the desired conclusion. \[\square\]