Math 341 Lecture #28
§6.2: Uniform Convergence

Recall that we constructed a continuous nowhere differentiable function by way of a convergent series of continuous functions:

\[ g(x) = \sum_{n=0}^{\infty} h_n(x). \]

We showed that for each \( x \) the series converged, that the sequence of partials sums converged.

We give this “type” of convergence a name.

**Definition 6.2.1.** For each \( n \in \mathbb{N} \) let \( f_n : A \to \mathbb{R} \) for \( A \subseteq \mathbb{R} \). The sequence \( (f_n) \) converges pointwise on \( A \) to a function \( f : A \to \mathbb{R} \) if for all \( x \in A \) the sequence of real numbers \( f_n(x) \) converges to \( f(x) \).

Notations for this pointwise convergence of \( f_n \) to \( f \) on \( A \) are

\[ f_n \to f, \quad \lim f_n = f, \quad \lim_{n \to \infty} f_n(x) = f(x), \]

where the domain \( A \), not written, is understood.

**Example 6.2.2.** (ii) For \( n \in \mathbb{N} \), let \( g_n(x) = x^n \) on the domain \( A = [0, 1] \).

Here are the graphs of \( g_1, g_2, g_3, g_4 \).

We notice that \( g_n(1) = 1 \) for all \( n \in \mathbb{N} \), so that

\[ \lim_{n \to \infty} g_n(1) = 1. \]

For \( 0 \leq x < 1 \), we have that

\[ \lim_{n \to \infty} g_n(x) = 0. \]
Although each $g_n$ is differentiable (and hence continuous) on $A$, the limit function

$$g(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1, \\ 1 & \text{if } x = 1, \end{cases}$$

is not continuous on $A$.

We have $g_n \to g$ pointwise: this sequence of continuous functions converges to a discontinuous function.

This says that pointwise convergence of a series of continuous functions is not enough to guarantee the limit function if continuous; we need a stronger “type” of convergence.

**Definition 6.2.3.** Let $f_n$ be a sequence of functions defined on $A \subseteq \mathbb{R}$. We say that $(f_n)$ converges uniformly on $A$ to a limit function $f$ on $A$ if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \epsilon$ for all $x \in A$ whenever $n \geq N$.

Certainly, uniform convergence implies pointwise convergence, but the converse is false (as we have seen), so that uniform convergence is a stronger “type” of convergence than pointwise convergence.

**Example. 6.2.4.** (i) Let

$$g_n(x) = \frac{1}{n(1 + x^2)}.$$ 

For any $x \in \mathbb{R}$, we see that $g_n(x) \to 0$, so that $g_n$ converges pointwise to $g(x) = 0$.

Is this convergence uniform on $\mathbb{R}$?

Well, since $1/(x^2 + 1) \leq 1$ for all $x \in \mathbb{R}$, we have

$$|g_n(x) - g(x)| = \left| \frac{1}{n(x^2 + 1)} - 0 \right| \leq \frac{1}{n}.$$ 

Thus for a given $\epsilon > 0$ we can choose $N \geq 1/\epsilon$, a choice of $N$ that is independent of $x \in \mathbb{R}$, so that $|g_n(x) - g(x)| < \epsilon$ for all $x \in \mathbb{R}$ whenever $n \geq N$.

This says that $g_n$ converges uniformly to $g$ on $\mathbb{R}$.

(ii) It is “easy” to guess that the sequence of functions

$$f_n(x) = \frac{x^2 + nx}{n}$$

converges pointwise to $f(x) = x$ on all of $\mathbb{R}$.

Investigating whether this convergence is uniform, we consider

$$|f_n(x) - f(x)| = \left| \frac{x^2 + nx}{n} - x \right| = \left| \frac{x^2 + nx - nx}{n} \right| = \frac{x^2}{n}.$$ 

Asking that $|f_n(x) - f(x)| < \epsilon$ for $\epsilon > 0$ requires that we choose $N \in \mathbb{N}$ according to

$$N > \frac{x^2}{\epsilon}.$$
This says that we cannot choose one value of $N$ that works for all $x \in \mathbb{R}$, and so the pointwise convergence of $f_n$ to $f$ is not uniform on $\mathbb{R}$.

If instead, we restrict the domain of $f_n$ to the compact subset $[-b, b]$ (for $b > 0$), then we can get uniform convergence of $f_n$ to $f$ by choosing

$$N > \frac{b^2}{\epsilon}.$$ 

That is we have for all $x \in [-b, b]$ that

$$|f_n(x) - f(x)| = \frac{x^2}{n} \leq \frac{b^2}{n} \leq \frac{b^2}{N} < \epsilon$$

for all $n \geq N$.

For $b = 1$, what does uniform convergence of $f_n$ to $f$ on $[-1, 1]$ means geometrically?

For $\epsilon = 1/2$ and $b = 1$ we have that $N = 3 > 1^2/(1/2) = 2$, and we get

$$|f_n(x) - f(x)| < 1/2 \text{ or } f(x) - 1/2 < f_n(x) < f(x) + 1/2$$

for all $x \in [-1, 1]$ and all $n \geq 3$.

Here are the graphs of $f(x) = x$, $f(x) + 1/2$, $f(x) - 1/2$, and $f_n(x)$ for $n = 3, 4, 5$.

What do you notice about the graphs of $f_n(x)$ for $n = 3, 4, 5$? Each lies within the $\epsilon = 1/2$ “tube” about $f(x)$ on $[-1, 1]$.

This is what uniform convergence means geometrically.

Recall the Cauchy Criterion for the convergence of a sequence of real numbers did not require a “guess” for the limit.

We have a similar criterion for uniform convergence.
Theorem 6.2.5 (Cauchy Criterion for Uniform Convergence). A sequence of functions \((f_n)\) defined on \(A \subseteq \mathbb{R}\) converges uniformly on \(A\) if and only if for every \(\epsilon > 0\) there exists an \(N \in \mathbb{N}\) such that

\[
|f_n(x) - f_m(x)| < \epsilon
\]

for all \(n, m \geq N\) and all \(x \in A\).

Proof. Suppose that \(f_n\) converges uniformly to \(f\) on \(A\).

Then for \(\epsilon > 0\) there exists \(N \in \mathbb{N}\) such that \(|f_n(x) - f(x)| < \epsilon/2\) for all \(n \geq N\) and all \(x \in A\).

Thus for \(n, m \geq N\) and any \(x \in A\) we have

\[
|f_n(x) - f_m(x)| = |f_n(x) - f(x) + f(x) - f_m(x)| \\
\leq |f_n(x) - f(x)| + |f_m(x) - f(x)| \\
< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
= \epsilon.
\]

Now suppose that for all \(\epsilon > 0\) there exists \(N \in \mathbb{N}\) such that \(|f_n(x) - f_m(x)| < \epsilon\) for all \(n, m \geq N\) and all \(x \in A\).

Then for each \(x \in A\), the sequence \((f_n(x))\) is Cauchy sequence of real numbers, and therefore it converges to a real number, call it \(f(x)\).

We have found a function \(f : A \to \mathbb{R}\) which is the pointwise limit of \(f_n\).

By the Algebraic Limit and Order Limit Theorems we have for all \(n \geq N\) and all \(x \in A\) that

\[
|f_n(x) - f(x)| = \lim_{m \to \infty} |f_n(x) - f_m(x)| \leq \epsilon
\]

which says that \(f_n\) converges uniformly to \(f\) on \(A\). \(\square\)

The stronger assumption of uniform convergence is enough to guarantee that the limit function of a sequence of continuous functions is continuous.

Theorem 6.2.6 (Continuous Limit Theorem). Let \((f_n)\) be a sequence of functions defined on \(A \subseteq \mathbb{R}\) that converges uniformly on \(A\) to \(f\). If each \(f_n\) is continuous at \(c \in A\), then \(f\) is continuous at \(c\) too.

Proof. Fix \(c \in A\), and for \(\epsilon > 0\) choose \(N \in \mathbb{N}\) such that for all \(x \in A\) we have

\[
|f_N(x) - f(x)| < \frac{\epsilon}{3}.
\]

By the continuity of \(f_N\) at \(c\) there exists \(\delta > 0\) such that whenever \(|x - c| < \delta\) we have

\[
|f_N(x) - f_N(c)| < \frac{\epsilon}{3}.
\]

Thus

\[
|f(x) - f(c)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(c)| + |f_N(c) - f(c)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon
\]

whenever \(|x - c| < \delta\), and so \(f\) is continuous at \(c\). \(\square\)