Math 341 Lecture #33
§7.2: The Definition of the Riemann Integral

We begin by recalling the definition of the Riemann integral.

Starting with a function $f$ on $[a, b]$, we partition $[a, b]$ into subintervals $[x_{k-1}, x_k]$, pick sample points $c_k \in [x_{k-1}, x_k]$, and form the sum

$$
\sum_{k=1}^{n} f(c_k) \Delta x_k
$$

where $\Delta x_k = x_k - x_{k-1}$.

We defined the integral of $f$ on $[a, b]$ to be the limit of the Riemann sums as $\max \Delta_k \to 0$, provided the limit exists.

We will approach the Riemann integral differently, using supremums and infimums.

Let $f$ be a bounded function defined on the compact interval $[a, b]$; there is $M > 0$ such that $|f(x)| \leq M$ for all $x \in [a, b]$.

Definition 7.2.1. A partition of $[a, b]$ is a finite, ordered set of points

$$
P = \{ a = x_0 < x_1 < x_2 < \cdots < x_n = b \}.
$$

Set $\Delta x_k = x_k - x_{k-1}$ for each $k = 1, 2, \ldots, n$.

For each subinterval $[x_{k-1}, x_k]$ of $P$ we set

$$
m_k = \inf\{ f(x) : x \in [x_{k-1}, x_k] \} \quad \text{and} \quad M_k = \sup\{ f(x) : x \in [x_{k-1}, x_k] \}.
$$

The lower sum of $f$ with respect to $P$ is given by

$$
L(f, P) = \sum_{k=1}^{n} m_k \Delta x_k.
$$

The upper sum of $f$ with respect to $P$ is given by

$$
U(f, P) = \sum_{k=1}^{n} M_k \Delta x_k.
$$

For a given partition, because $m_k \leq M_k$ on each $[x_{k-1}, x_k]$ we have that

$$
L(f, P) \leq U(f, P).
$$

But what happens when we change partitions?

Definition 7.2.2. A partition $Q$ is a refinement of a partition $P$, written $P \subseteq Q$, if $Q$ contains all of the points of $P$.

Lemma 7.2.3. If $P \subseteq Q$, then $L(f, P) \leq L(f, Q)$ and $U(f, P) \geq U(f, Q)$.

Proof. Let $[x_{k-1}, x_k]$ be an subinterval of $P$. 
Suppose there is a point \( z \in \mathbb{Q} \) such that \( x_{k-1} < z < x_k \), so that \( z \) “splits” \([x_{k-1}, x_k]\) into two subintervals \([x_{k-1}, z]\) and \([z, x_k]\).

We have the infimum \( m_k \) of \( f \) on \([x_{k-1}, x_k]\).

Let \( m'_k = \inf \{ f(x) : x \in [z, x_k] \} \) and \( m''_k = \inf \{ f(x) : x \in [x_{k-1}, z] \} \).

Then we have that \( m_k \leq m'_k \) and \( m_k \leq m''_k \) (the infimum of \( f \) over a larger set cannot get bigger), so that

\[
m_k \Delta x_k = m_k(x_k - x_{k-1})
= m_k(x_k - z + z - x_{k-1})
= m_k(x_k - z) + m_k(z - x_{k-1})
\leq m'_k(x_k - z) + m''_k(z - x_k).
\]

This implies that the lower sum cannot get smaller when adding more points to a partition.

A similar argument show that the upper sum cannot get bigger when adding more points to a partition. \( \square \)

We would think it strange if a lower sum for one partition were bigger than an upper sum for another partition. Luckily this cannot happen.

Lemma 7.2.4. If \( P_1 \) and \( P_2 \) are partitions of \([a, b]\), then \( L(f, P_1) \leq U(f, P_2) \).

Proof. We form a third partition and use Lemma 7.2.3.

The partition \( Q = P_1 \cup P_2 \) is a refinement of \( P_1 \) and a refinement of \( P_2 \); it is a common refinement of both \( P_1 \) and \( P_2 \).

Because \( P_1 \subseteq Q \) and \( P_2 \subseteq Q \), we have that

\[
L(f, P_1) \leq L(f, Q) \leq U(f, Q) \leq U(f, P_2)
\]

by Lemma 7.2.3. \( \square \)

This raises the questions: what is the supremum of the lower sums, and what is the infimum of the upper sums, and are they the same?

Definition 7.2.5. Let \( \mathcal{P} \) be the collection of all possible partitions of \([a, b]\). The upper integral of \( f \) on \([a, b]\) is

\[
U(f) = \inf \{ U(f, P) : P \in \mathcal{P} \}
\]

and the lower sum of \( f \) is

\[
L(f) = \sup \{ L(f, P) : P \in \mathcal{P} \}.
\]

Lemma 7.2.6. For any bounded function \( f \) on \([a, b]\), there holds \( L(f) \leq U(f) \).

Proof. For partitions \( P \) and \( Q \) of \([a, b]\) we have by Lemma 7.2.4 that \( L(f, P) \leq U(f, Q) \).

So \( U(f, Q) \) is an upper bound on the set \( \{ L(f, P) : P \in \mathcal{P} \} \).

It follows that

\[
L(f) = \sup \{ L(f, P) : P \in \mathcal{P} \} \leq U(f, Q).
\]
This says that \( L(f) \) is a lower bound for \( U(f, Q) \), so that
\[
L(f) \leq \inf \{ U(f, Q) : Q \in \mathcal{P} \} = U(f),
\]
thus giving the inequality. \( \square \)

A special situation happens when \( L(f) = U(f) \).

Definition 7.2.7 (Riemann Integrability) A bounded function \( f \) on the compact interval \([a, b]\) if Riemann-integrable if \( U(f) = L(f) \). In this case we write
\[
\int_a^b f = U(f) = L(f).
\]

There are other types of integration besides the Riemann integral, but we will stick to the Riemann integral here, and drop the “Riemann” from Riemann-integrable.

What bounded functions on \([a, b]\) are integrable?

We have the following integrability criterion, which in part will identity which bounded functions are integrable. [The complete answer is in Section 7.6 at the top of page 242.]

Theorem 7.2.8. A bounded function \( f \) is integrable on \([a, b]\) if and only if for every \( \epsilon > 0 \) there exists a partition \( P_\epsilon \) of \([a, b]\) such that
\[
U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon.
\]

Proof. Suppose for each \( \epsilon > 0 \) there exists a partition \( P_\epsilon \) of \([a, b]\) such that \( U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon \).

Since \( U(f) \leq U(f, P_\epsilon) \) and \( L(f) \geq L(f, P_\epsilon) \) we have that
\[
U(f) - L(f) \leq U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon.
\]
The arbitrariness of \( \epsilon > 0 \) then implies that \( U(f) = L(f) \), and so \( f \) is integrable.

Now suppose that \( f \) is integrable so that \( U(f) = L(f) \).

Let \( \epsilon > 0 \).

Since \( U(f) \) is the infimum of \( U(f, P) \) over all partitions \( P \) of \([a, b]\), there exists a partition \( P_1 \) such that
\[
U(f, P_1) < U(f) + \frac{\epsilon}{2}.
\]

Since \( L(f) \) is the supremum of \( L(f, P) \) over all partitions \( P \) of \([a, b]\), there exists a partition \( P_2 \) such that
\[
L(f, P_2) > L(f) - \frac{\epsilon}{2}.
\]

For the common refinement \( P_\epsilon = P_1 \cup P_2 \) and with \( U(f) = L(F) \) we have
\[
U(f, P_\epsilon) - L(f, P_\epsilon) \leq U(f, P_1) - L(f, P_2)
= U(f, P_1) - U(f) + L(f) - L(f, P_2)
< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,
\]
which gives the integrability criterion.

A continuous function \( f \) on the compact \([a, b]\) is bounded by the Extreme Value Theorem.

We are now in position to prove that every continuous function on \([a, b]\) is integrable.

**Theorem 7.2.9.** If \( f \) is continuous on the compact \([a, b]\), then \( f \) is integrable.

**Proof.** The continuity of \( f \) on the compact \([a, b]\) implies the uniform continuity of \( f \) on \([a, b]\).

Thus for \( \epsilon > 0 \) there exists \( \delta > 0 \) such that when \( |x - y| < \delta \) with \( x, y \in [a, b] \), we have

\[
|f(x) - f(y)| < \frac{\epsilon}{b - a}.
\]

We are driving for the criterion for integrability.

To this end we choose a partition \( P \) of \([a, b]\) where \( \Delta x_k < \delta \) for all \( k = 1, \ldots, n \).

The function \( f \) is continuous on the compact subinterval \([x_{k-1}, x_k]\) of \( P \), so we have by the Extreme Value Theorem points \( y_k, z_k \in [x_{k-1}, x_k] \) where \( f(y_k) = m_k \) and \( f(z_k) = M_k \).

Since \( y_k, z_k \in [x_{k-1}, x_k] \) we have that

\[
M_k - m_k = f(z_k) - f(y_k) < \frac{\epsilon}{b - a}.
\]

This implies that

\[
U(f, P) - L(f, P) = \sum_{k=1}^{n} (M_k - m_k) \Delta x_k < \frac{\epsilon}{b - a} \sum_{k=1}^{n} \Delta x_k = \epsilon.
\]

Therefore, \( f \) is integrable by Theorem 7.2.8.

This raises the question about how discontinuous a bounded function can be and yet still be integrable.