Lebesgue’s Monotone Convergence Theorem (1.26). If $f_n : X \to [0, \infty]$ is a sequence of monotonic nondecreasing measurable functions, then \( \{f_n\} \) is pointwise convergent, the pointwise limit function \( f = \lim_{n \to \infty} f_n \) is measurable, and

$$
\lim_{n \to \infty} \int_X f_n d\mu = \int_X f d\mu.
$$

Proof: The nondecreasing monotonicity of \( \{f_n\} \) implies that

$$
\lim_{n \to \infty} f_n = \sup_n f_n.
$$

Thus \( \{f_n\} \) is pointwise convergent, and, by Theorem 1.14, that \( f = \lim_{n \to \infty} f_n \) is measurable.

Since \( f_n \leq f_{n+1} \), then

$$
\int_X f_n d\mu \leq \int_X f_{n+1} d\mu,
$$

so that there is a unique \( \alpha \in [0, \infty] \) for which

$$
\int_X f_n d\mu \to \alpha \text{ as } n \to \infty.
$$

Since \( f_n \leq f \), then

$$
\int_X f_n d\mu \leq \int_X f d\mu,
$$

so that

$$
\alpha = \lim_{n \to \infty} \int_X f_n d\mu \leq \int_X f d\mu.
$$

We will show that the opposite inequality holds as well.

For a simple measurable \( s : X \to [0, \infty) \) such that \( 0 \leq s \leq f \), and a constant \( c \in (0, 1) \), define

$$
E_n = \{ x : f_n(x) \geq cs(x) \}.
$$

Then each \( E_n \) is measurable [by homework problem Ch1. #5(a)].

Also, \( E_1 \subset E_2 \subset E_3 \subset \cdots \) because \( f_n \leq f_{n+1} \).

Furthermore, \( X = \bigcup_{n=1}^{\infty} E_n \) because if \( f(x) = 0 \) then \( cs(x) \leq s(x) \leq f(x) = f_n(x) = 0 \) for any \( n \), so that \( x \in E_1 \); and if \( f(x) > 0 \), then \( cs(x) < f_n(x) \leq f(x) (0 \leq cs < s \leq f) \) for all large enough \( n \), so that \( x \in E_n \) for some \( n \).

Thus

$$
\int_X f_n d\mu \geq \int_{E_n} f_n d\mu \geq c \int_{E_n} s d\mu = c \varphi(E_n).
$$
By Proposition 1.25, \( \varphi \) is a positive measure, and hence
\[
\alpha = \lim_{n \to \infty} \int_X f_n d\mu \\
\geq \lim_{n \to \infty} c\varphi(E_n) \\
= c\varphi(X) \quad [X = \bigcup E_n, E_n \subset E_{n+1}, \text{“continuity” of } \mu] \\
= c \int_X s d\mu.
\]
Since \( 0 < c < 1 \), then
\[
\alpha \geq \int_X s d\mu.
\]
This holds for every simple measurable \( s \) such that \( 0 \leq s \leq f \).
Therefore
\[
\alpha \geq \int_X f d\mu,
\]
which completes the proof. \( \square \)

Fatou’s Lemma (1.28). If \( f_n : X \to [0, \infty] \) is a sequence of measurable functions, then
\[
\int_X \left( \liminf_{n \to \infty} f_n \right) d\mu \leq \liminf_{n \to \infty} \int_X f_n d\mu.
\]

Proof: Set \( g_n = \inf_{k \geq n} f_k \).
Then \( \{g_n\} \) is a monotone nondecreasing sequence of measurable functions such that \( g_n \leq f_n \).
This implies that
\[
\lim_{n \to \infty} \int_X g_n d\mu = \sup_n \int_X g_n d\mu = \liminf_{n \to \infty} \int_X g_n d\mu.
\]

\[
\lim_{n \to \infty} g_n = \lim_{n \to \infty} \left\{ \inf_{k \geq n} f_k \right\} \\
= \sup_n \left\{ \inf_{k \geq n} f_k \right\} \\
= \liminf_{n \to \infty} f_n,
\]
and
\[
\int_X g_n d\mu \leq \int_X f_n d\mu.
\]
The last implies that
\[
\liminf_{n \to \infty} \int_X g_n d\mu \leq \liminf_{n \to \infty} \int_X f_n d\mu.
\]
It follows by Lebesgue’s Monotone Convergence Theorem that
\[
\liminf_{n \to \infty} \int_X f_n d\mu \geq \liminf_{n \to \infty} \int_X g_n d\mu \quad \text{[above inequality]}
\]
\[
= \lim_{n \to \infty} \int_X g_n d\mu \quad \text{[monotonicity of \(\left\{ \int_X g_n d\mu \right\}\)]}
\]
\[
= \int_X \lim_{n \to \infty} g_n d\mu \quad \text{[LMCT]}
\]
\[
= \int_X \liminf_{n \to \infty} f_n d\mu, \quad \text{[calculation above]}
\]
which completes the proof. □

Proposition (1.25, 2nd half). If \(s : X \to [0, \infty]\) and \(t : X \to [0, \infty]\) are simple measurable functions, then
\[
\int_X (s + t) d\mu = \int_X s d\mu + \int_X t d\mu.
\]

Theorem (1.27). If \(f_n : X \to [0, \infty]\) is a sequence of measurable functions, then the series \(\sum_{n=1}^{\infty} f_n\) is pointwise convergent (in the sense of the partial sums), the pointwise limit function is measurable, and
\[
\int_X \sum_{n=1}^{\infty} f_n d\mu = \sum_{n=1}^{\infty} \int_X f_n d\mu.
\]

Theorem (1.29 first part): If \(f : X \to [0, \infty]\) is measurable, then \(\varphi : \mathcal{M} \to [0, \infty]\) defined by
\[
\varphi(E) = \int_E f d\mu
\]
is a positive measure on \(\mathcal{M}\).

Corollary (1.29 second part). If \(\varphi\) is the positive measure on \(\mathcal{M}\) “induced” by a measurable \(f : X \to [0, \infty]\) (as in Theorem 1.29), then
\[
\int_X g d\varphi = \int_X g f d\mu
\]
for every measurable \(g : X \to [0, \infty]\).

Remark. This Corollary says that
\[
d\varphi = f d\mu \quad \text{or} \quad \frac{d\varphi}{d\mu} = f.
\]

If \(\int_X f d\mu < \infty\), then \(f\) is known as the **Radon-Nikodym derivative** of \(\varphi\) with respect to \(\mu\).

More about this in Chapter 6.
Appendix Proofs.

Proof of Proposition 1.25 (second part): Suppose $s = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$ and $t = \sum_{j=1}^{m} \beta_j \chi_{B_j}$ are measurable.

Then the finitely many sets $E_{ij} = A_i \cap B_j$ are measurable, pairwise disjoint, and satisfy $X = \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} E_{ij}$.

Moreover, $(s + t) \chi_{E_{ij}} = (\alpha_i + \beta_j) \chi_{E_{ij}}$ (uses $\chi_A \chi_B = \chi_{A \cap B}$), and

$$s + t = \sum_{i=1}^{n} \sum_{j=1}^{m} (\alpha_i + \beta_j) \chi_{E_{ij}}.$$

[Each $x \in X$ belongs to exactly one $E_{ij}$, for which $(s + t)(x) = \alpha_i + \beta_j$.]

Using Proposition 1.25 (first part), define three positive measures on $\mathcal{M}$ by

$$\varphi(E) = \int_E s d\mu, \quad \psi(E) = \int_E t d\mu, \quad \theta(E) = \int_E (s + t) d\mu.$$

Then

$$\theta(E_{ij}) = \int_{E_{ij}} (s + t) d\mu$$
$$= \int_X (s + t) \chi_{E_{ij}} d\mu = \int_X (\alpha_i + \beta_j) \chi_{E_{ij}} d\mu$$
$$= (\alpha_i + \beta_j)\mu(E_{ij}) = \alpha_i \mu(E_{ij}) + \beta_j \mu(E_{ij}) = \int_X \alpha_i \chi_{E_{ij}} d\mu + \int_X \beta_j \chi_{E_{ij}} d\mu$$
$$= \int_{E_{ij}} s d\mu + \int_{E_{ij}} t d\mu \quad [\chi_{B_j} \chi_{E_{ij}} = \chi_{E_{ij}} \text{ because } E_{ij} \subset B_j]$$
$$= \varphi(E_{ij}) + \psi(E_{ij}).$$

Therefore, by finite additivity,

$$\int_X (s + t) d\mu = \theta(X) = \theta \left( \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} E_{ij} \right) = \sum_{i=1}^{n} \sum_{j=1}^{m} \theta(E_{ij})$$
$$= \sum_{i=1}^{n} \sum_{j=1}^{m} \{ \varphi(E_{ij}) + \psi(E_{ij}) \}$$
$$= \sum_{i=1}^{n} \sum_{j=1}^{m} \varphi(E_{ij}) + \sum_{i=1}^{n} \sum_{j=1}^{m} \psi(E_{ij})$$
$$= \varphi \left( \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} E_{ij} \right) + \psi \left( \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} E_{ij} \right)$$
$$= \varphi(X) + \psi(X)$$
$$= \int_X s d\mu + \int_X t d\mu.$$

This completes the proof. \qed
Proof of Theorem 1.27. Set \( g_n = \sum_{i=1}^{n} f_i \). Then \( \{g_n\} \) is a monotone nondecreasing sequence of measurable functions.

By Lebesgue’s Monotone Convergence Theorem, \( \{g_n\} \) (the sequence of partial sums) converges pointwise, the pointwise limit function, \( g = \sum_{i=1}^{\infty} f_i \), is measurable, and

\[
\lim_{n \to \infty} \int_X (f_1 + \cdots + f_n) d\mu = \lim_{n \to \infty} \int_X g_n d\mu = \int_X \sum_{i=1}^{\infty} f_i d\mu.
\]

It remains to show that

\[
(\text{see boxed integral above}) \quad \int_X \sum_{i=1}^{n} f_i d\mu = \sum_{i=1}^{n} \int_X f_i d\mu,
\]

i.e. that we can switch the integral and finite sum.

By Theorem 1.17, there are monotone nondecreasing sequences \( \{s_n\}, \{t_n\} \) (where \( s_n, t_n : X \to [0, \infty) \)) of simple measurable functions such that

1. \( 0 \leq s_n \leq f_1 \) and \( s_n \to f_1 \) pointwise, and
2. \( 0 \leq t_n \leq f_2 \) and \( t_n \to f_2 \) pointwise.

Set \( u_n = s_n + t_n \).

Then \( \{u_n\} \) is monotone nondecreasing, \( 0 \leq u_n \leq f_1 + f_2 \), and \( u_n \to f_1 + f_2 \) pointwise: \( \sup_n \{s_n + t_n\} = f_1 + f_2 \).

By Lebesgue’s Monotone Convergence Theorem,

\[
\lim_{n \to \infty} \int_X u_n d\mu = \int_X \lim_{n \to \infty} u_n d\mu = \int_X (f_1 + f_2) d\mu.
\]

On the other hand, Proposition 1.25 (2nd half) and Lebesgue’s Monotone Convergence Theorem imply that

\[
\lim_{n \to \infty} \int_X u_n d\mu = \lim_{n \to \infty} \int_X (s_n + t_n) d\mu
\]

\[
= \lim_{n \to \infty} \left\{ \int_X s_n d\mu + \int_X t_n d\mu \right\} \quad \text{Proposition 1.25, 2nd part}
\]

\[
= \lim_{n \to \infty} \int_X s_n d\mu + \lim_{n \to \infty} \int_X t_n d\mu
\]

\[
= \int_X f_1 d\mu + \int_X f_2 d\mu. \quad \text{LMCT (twice)}
\]

Induction then gives

\[
\int_X \sum_{i=1}^{n} f_i d\mu = \sum_{i=1}^{n} \int_X f_i d\mu.
\]

This completes the proof. \( \square \)
Proof of Theorem 1.29 (first part). There is an \( E \in \mathcal{M} \) such that \( \varphi(E) < \infty \), namely \( E = \emptyset \) for which
\[
\varphi(\emptyset) = \int_0 \, f \, d\mu = 0
\]
because \( \mu(\emptyset) = 0 \).

Suppose \( E_1, E_2, E_3, \ldots \in \mathcal{M} \) is a countable collection of pairwise disjoint sets.

Set \( E = \bigcup_{n=1}^{\infty} E_n \).

Then \( E \in \mathcal{M} \), and
\[
\chi_E f = \sum_{i=1}^{\infty} \chi_{E_i} f.
\]

By Theorem 1.27,
\[
\varphi \left( \bigcup_{i=1}^{\infty} E_i \right) = \varphi(E) = \int_E f \, d\mu
\]
\[
= \int_X \chi_E f \, d\mu = \int_X \sum_{i=1}^{\infty} \chi_{E_i} f \, d\mu
\]
\[
= \sum_{i=1}^{\infty} \int_X \chi_{E_i} f \, d\mu = \sum_{i=1}^{\infty} \int_{E_i} f \, d\mu = \sum_{i=1}^{\infty} \varphi(E_i).
\]

Thus \( \varphi \) is countable additive. \( \square \)

Proof of the Corollary. For \( g = \chi_E \) with \( E \in \mathcal{M} \), Theorem 1.29 (first part) shows that
\[
\int_X g \, d\varphi = \int_X \chi_E d\varphi = \int_E d\varphi = \varphi(E) = \int_E f \, d\mu = \int_X \chi_E f \, d\mu = \int_X g \, f \, d\mu.
\]

Hence the Corollary holds for every simple measurable function \( g \).

For an arbitrary measurable function \( g \) there is by Theorem 1.17 a sequence of simple measurable functions \( \{ s_n \} \) such that \( 0 \leq s_1 \leq s_2 \leq \ldots \) and \( s_n \to g \).

Then by two applications of Lesbesgue’s Monotone Convergence Theorem, we have
\[
\int_X g \, d\varphi = \lim_{n \to \infty} \int_X s_n \, d\varphi = \lim_{n \to \infty} \int_X s_n \, f \, d\mu = \int_X g \, f \, d\mu.
\]

This completes the proof. \( \square \)