Complex Measures and Integration

Definition (6.18). Let \( \mu \) be a complex measure on a \( \sigma \)-algebra \( \mathcal{M} \) in \( X \).

There is (by Theorem 6.12) a measurable function \( h \) such that \(|h| = 1 \) and \( d\mu = hd|\mu| \) [actually, \( h \) is a \( L^1(|\mu|) \) function that depends on \( \mu \)].

The integral of a measurable function \( f \) on \( X \) with respect to \( \mu \) is

\[
\int_X f \, d\mu = \int_X fh \, d|\mu|.
\]

A special case of this is

\[
\int_X \chi_E \, d\mu = \int_X \chi_E h \, d|\mu| = \int_E h \, d|\mu| = \mu(E),
\]

which justifies setting \( \mu(E) = \int_E \, d\mu \) for all \( E \in \mathcal{M} \).

The Dual of \( C_0(X) \)

Definition. Let \( M(X) \) be the collection of all regular complex Borel measures on a LCH space \( X \), and equip \( M(X) \) with the total variation norm, \( \|\mu\| = |\mu|(X) \).

[RECALL: a regular complex Borel measure is a complex measure \( \mu \) on \( \mathcal{B}_X \) such that \(|\mu| \) is regular.]

Proposition. \( M(X) \) is a Banach space.

Proof. Homework Problem Ch.6 #3.

Problem. Each \( \mu \in M(X) \) defines a linear functional \( I_\mu : C_0(X) \to \mathbb{C} \) by

\[
I_\mu(f) = \int_X f \, d\mu,
\]

where (for the unique measurable \( h \) corresponding to \( \mu \))

\[
|I_\mu(f)| = \left| \int_X f \, d\mu \right| = \left| \int_X fh \, d|\mu| \right| \leq \int_X |fh| \, d|\mu| \leq \|f\|_u \int_X 1 \, d|\mu| \leq \|f\|_u |\mu|(X).
\]

Hence \( \|I_\mu\| \leq \|\mu\| < \infty \), so that \( I_\mu \in C_0(X)^* \).

Does \( C_0(X)^* = \{I_\mu : \mu \in M(X)\} \)?
Define a map $I : M(X) \to C_0(X)^*$ by $I : \mu \to I\mu$.

Proposition. The map $I$ is linear.

Proof. For $c \in \mathbb{C}$ and $\mu, \lambda \in M(X)$ and $f = \chi_E$ ($E \in \mathcal{M}$),
\[
I_{c\mu}(f) = \int_X \chi_E \, d(c\mu) = (c\mu)(E) = c \mu(E) = c \int_X \chi_E \, d\mu = cI\mu(f),
\]
and
\[
I_{\mu+\lambda}(f) = \int_X \chi_E \, d(\mu+\lambda) = (\mu+\lambda)(E) = \mu(E) + \lambda(E)
\]
\[
= \int_X \chi_E \, d\mu + \int_X \chi_E \, d\lambda = I\mu(f) + I\lambda(f).
\]
By linearity of integration, this extends to all simple Borel functions $f$, and hence to all bounded Borel functions, which contains $C_0(X)$.

Riesz Representation Theorem for $C_0(X)^*$ (6.19). If $X$ is LCH, then $I : M(X) \to C_0(X)^*$ is a isometric isomorphism.

Proof. Let $\Phi \in C_0(X)^*$.

(Injectiveness)
Suppose there are $\mu_1, \mu_2 \in M(X)$ such that $I_{\mu_1} = \Phi = I_{\mu_2}$.
For $\mu = \mu_2 - \mu_1 \in M(X)$ there is a (unique) Borel function $h \in L^1(|\mu|)$ such that
\[
0 = I_{\mu_2}(f) - I_{\mu_1}(f) = I_{\mu_2-\mu_1}(f) = \int_X f \, d(\mu_2 - \mu_1) = \int_X f \, d\mu = \int_X fh \, d|\mu|
\]
for all $f \in C_0(X)$.
Hence for any sequence $\{f_n\} \subset C_0(X)$,
\[
|\mu|(X) = \int_X d|\mu| = \int_X |h|^2 d|\mu| = \int_X \bar{h}h \, d|\mu| - 0
\]
\[
= \int_X \bar{h}h \, d|\mu| - \int_X f_n h \, d|\mu|
\]
\[
= \int_X (\bar{h} - f_n)h \, d|\mu| \leq \int_X |\bar{h} - f_n| \, d|\mu| = ||\bar{h} - f_n||_1.
\]
Since $C_c(X)$ is dense in $L^1(|\mu|)$ (Theorem 3.14), there is a sequence $\{f_n\} \subset C_c(X)$ such that $||\bar{h} - f_n||_1 \to 0$.
The inclusion $C_c(X) \subset C_0(X)$ implies that $|\mu|(X) = 0$; hence $\mu = 0$.

(Surjectiveness and Isometricness)
Assume that $||\Phi|| = \sup\{|\Phi(f)| : ||f||_u = 1, f \in C_0(X)\} = 1$ [rescale $\Phi$ when $||\Phi|| > 0$; choose $\mu = 0$ when $||\Phi|| = 0$].
Suppose there is a positive linear functional $\Lambda$ on $C_c(X)$ such that

$$|\Phi(f)| \leq \Lambda(|f|) \leq \|f\|_u$$

for all $f \in C_c(X)$. [The existence of $\Lambda$ will be shown in the next Lecture.]

Then by the Riesz Representation Theorem for Positive Linear Functionals on $C_c(X)$ (Theorem 2.14), there is a unique positive Borel measure $\lambda$ such that

$$\Lambda(f) = \int_X f \, d\lambda$$

where $\lambda$ is outer regular on all $B_X$ and inner regular on every open $E \subset X$ or any $E \in M$ with $\lambda(X) < \infty$ [and $\lambda(K) < \infty$ for all compact $K \subset X$; i.e. $\lambda$ is a Radon measure].

Since $\lambda(V) = \sup\{\Lambda f : 0 \leq f \leq 1, f|_{C_c} = 0, f \in C_c(X)\}$ for $V$ open (this is the definition of $\lambda$ in Theorem 2.14), and since $X$ is open,

$$\lambda(X) = \sup\{\Lambda f : 0 \leq f \leq 1, f \in C_c(X)\} \leq \sup\{\|f\|_u : 0 \leq f \leq 1, f \in C_c(X)\} = 1.$$ 

Thus $\lambda(E) \leq \lambda(X) = 1$ for all $E \in B_X$, and so $\lambda$ is regular by Corollary A in Lecture #11.

On the subspace $C_c(X)$ of $L^1(\lambda)$, the functional $\Phi$ has norm at most 1:

$$|\Phi(f)| \leq \Lambda(|f|) = \int_X |f| \, d\lambda = \|f\|_1.$$ 

By the Hahn-Banach Theorem, there is a norm-preserving extension of $\Phi$ to $L^1(\lambda)$; call it $\Phi$.

By Theorem 6.16 (the Representation Theorem for $L^p(\mu)^*$ when $1 \leq p < \infty$), there is a $g \in L^\infty(\lambda)$ such that $\|g\|_\infty = \|\Phi\| = 1$ and

$$\Phi(f) = \int_X fg \, d\lambda$$

for all $f \in L^1(\lambda)$.

Since each side of this is continuous (with respect to the uniform norm) on $C_c(X)$, and since $C_c(X)$ is dense in $C_0(X)$ (in the uniform norm, Theorem 3.17), it follows that $\Phi(f) = \int_X fg \, d\lambda$ for all $f \in C_0(X)$.

Since $\int_X |fg| \, d\lambda \geq |\int_X fg \, d\lambda| = |\Phi(f)|$, it follows that

$$\int_X |g| \, d\lambda = \sup \left\{ \int_X |fg| \, d\lambda : f \in C_0(X), \|f\|_u \leq 1 \right\} \geq \|\Phi\| = 1.$$ 

This together with $\lambda(X) \leq 1$ and $|g| \leq \|g\|_\infty = 1$ a.e.$[\lambda]$ implies that $\lambda(X) = 1$ and $|g| = 1$ a.e.$[\lambda]$.

The complex Borel measure $\mu$ defined by $d\mu = gd\lambda$ satisfies $d|\mu| = |g|d|\lambda| = d\lambda$ (Theorem 6.13) and $\Phi(f) = \int_X f \, d\mu$ for all $f \in C_0(X)$; also $\mu$ is regular (since $\lambda$ is regular), and $\|\mu\| = |\mu|(X) = \lambda(X) = 1 = \|\Phi\| = \|I_\mu\|$.

Thus $I$ is an surjective linear isometry, and hence an isometric isomorphism.