Derivatives of Measures

Definition (7.2). The symmetric derivative of a complex Borel measure $\mu$ on $\mathbb{R}^k$ with respect to Lebesgue measure $m$ on $\mathbb{R}^k$ is the function $D\mu$ defined by

$$(D\mu)(x) = \lim_{r \to 0} \left( Q_r \mu \right)(x) = \lim_{r \to 0} \frac{\mu(B(x,r))}{m(B(x,r))}$$

at those $x$’s where the limit exists.

Theorem (7.8). If $\mu$ is a complex Borel measure on $\mathbb{R}^k$ such that $\mu \ll m$, then $D\mu = d\mu/dm$ a.e. $[m]$ (i.e. the Radon-Nikodym derivative of $\mu$ with respect to $m$) and

$$\mu(E) = \int_E (D\mu) \ dm \text{ for all } E \in B_{\mathbb{R}^k}.$$

Proof. If $f \in L^1(\mathbb{R}^k)$ is the Radon-Nikodym derivative of $\mu$ with respect to $m$, then

$$\mu(E) = \int_E f \ dm \text{ for all } E \in B_{\mathbb{R}^k}.$$

At any Lebesgue point $x$ of $f$,

$$\lim_{r \to 0} \frac{\mu(B(x,r))}{m(B(x,r))} = \lim_{r \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} f(y) \ dm(y) - f(x) + f(x)$$

$$= \lim_{r \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} f(y) \ dm(y)$$

$$- \lim_{r \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} f(x) \ dm(y) + f(x)$$

$$= \lim_{r \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} (f(y) - f(x)) \ dm(y) + f(x)$$

$$= f(x).$$

Thus $(D\mu)(x)$ exists and equals $f(x)$ at every Lebesgue point of $f$.

Hence by Theorem 7.7 (a.e. existence of Lebesgue points), $D\mu = f$ a.e $[m]$. □

Definition. The upper derivative of a complex Borel measure $\mu$ on $\mathbb{R}^k$ with respect to $m$ is the function $\bar{D}\mu : \mathbb{R}^k \to [0, \infty)$ defined by

$$(\bar{D}\mu)(x) = \lim_{n \to \infty} \sup_{0 < r < 1/n} \left( Q_r |\mu| \right)(x) = \lim_{n \to \infty} \sup_{0 < r < 1/n} \left( \frac{|\mu|(B(x,r))}{m(B(x,r))} \right).$$

[NOTE: this is defined for all $x$ since the term inside the brackets is nonincreasing and bounded below.]
Proposition. If $\mu$ is a complex Borel measure on $\mathbb{R}^k$, then

(a) $\bar{D}\mu$ is Borel measurable,

(b) $(\bar{D}\mu)(x) \leq (M\mu)(x)$ for all $x \in \mathbb{R}^k$, and

(c) $(D\mu)(x) = 0$ at an $x$ where $(\bar{D}\mu)(x) = 0$.

Proof. (a) For each $n \in \mathbb{N}$, the function

$$f_n(x) = \sup_{0<r<1/n} (Q_r|\mu|(x))$$

is lower semicontinuous (proof similar to that for lower semicontinuity of $M\mu$).

By part (c) of Theorem 1.12, $f_n$ is Borel measurable.

Since $f_n(x)$ converges pointwise to $\bar{D}\mu$, the limit function is Borel measurable (by Corollary (a) of Theorem 1.14).

(b) Recall that the maximal function of $\mu$ is

$$(M\mu)(x) = \sup_{0<r<\infty} (Q_r|\mu|(x)) = \sup_{0<r<\infty} \frac{|\mu|(B(x,r))}{m(B(x,r))}.$$ 

Since

$$\sup_{0<r<1/n} (Q_r|\mu|(x)) \leq \sup_{0<r<\infty} (Q_r|\mu|(x))$$

for all $n \in \mathbb{N}$, it follows that $(\bar{D}\mu)(x) \leq (M\mu)(x)$ for all $x$.

(c) The inequality $|\mu(B(x,r))| \leq |\mu|(B(x,r))$ implies that

$$|(Q_r\mu)(x)| = \frac{|\mu(B(x,r))|}{m(B(x,r))} \leq \frac{|\mu|(B(x,r))}{m(B(x,r))} = (Q_r|\mu|(x)).$$

Hence, for each $n \in \mathbb{N}$ and any $0 < s < 1/n$

$$|(Q_s\mu)(x)| \leq \sup_{0<r<1/n} (Q_r|\mu|(x)).$$

If $(\bar{D}\mu)(x) = 0$, then the RHS converges to 0, forcing the LHS to converge to zero, which implies that $(D\mu)(x) = 0$. \hfill \Box

Theorem. If $\mu$ is a complex Borel measure on $\mathbb{R}^k$ such that $\mu \perp m$, then $D\mu = 0$ a.e.[m].

Proof. Since $(D\mu)(x) = 0$ whenever $(\bar{D}\mu)(x) = 0$ (by part (c) of the Proposition), it suffices to show that $\bar{D}\mu = 0$ a.e.[m] with $\mu \geq 0$.

Suppose that $\mu$ is concentrated on a set $A$ of Lebesgue measure zero (i.e. $\mu \perp m$).

For $\epsilon > 0$, the regularity of $\mu$ (Theorem 2.18) shows that there is a compact $K \subset A$ with $m(K) = 0$ and $\mu(K) > \|\mu\| - \epsilon$.
Define Borel measures \( \mu_1(E) = \mu(K \cap E) \) and \( \mu_2 = \mu - \mu_1 \).

Since \( E = (K \cap E) \cup (E - K) \) disjointly for every \( E \in \mathcal{B}_{\mathbb{R}^k} \),

\[
\mu_2(E) = \mu(E) - \mu_1(E) = \mu(E \cap K) + \mu(E - K) - \mu(E \cap K) = \mu(E - K).
\]

Since \( -\mu(K) < -\|\mu\| + \epsilon \), it follows that

\[
\|\mu_2\| = \mu_2(\mathbb{R}^k) = \mu(\mathbb{R}^k - K) = \mu(K) - \|\mu\| < -\|\mu\| - \|\mu\| + \epsilon = \epsilon.
\]

For each \( x \notin K \), and sufficiently small \( r \), the intersection \( B(x, r) \cap K \) is empty. Hence \( \mu(B(x, r)) = \mu_1(B(x, r)) + \mu_2(B(x, r)) = 0 + \mu_2(B(x, r)) \), so that

\[
(\bar{D}\mu)(x) = (\bar{D}\mu_2)(x) \leq (M\mu_2)(x) \quad \text{for all } x \notin K,
\]

(the inequality with \( M\mu_2 \) by part (b) of the Proposition).

For each \( \lambda > 0 \),

\[
\{\bar{D}\mu > \lambda\} \subset K \cup \{M\mu_2 > \lambda\}
\]

since \( \mu \) is concentrated on \( A \) and \( \mu_2 \) is concentrated on \( A - K \).

Theorem 7.4 (the Maximal Theorem) gives \( m\{M\mu_2 > \lambda\} \leq 3^k \lambda^{-1}\|\mu_2\| \leq 3^k \lambda^{-1}\epsilon \).

Since \( m(K) = 0 \), it follows that \( m\{\bar{D}\mu > \lambda\} \leq 0 + 3^k \lambda^{-1}\epsilon \) [that \( \{\bar{D}\mu > \lambda\} \) is measurable follows from part (a) of the Proposition].

The arbitrariness of \( \epsilon > 0 \) implies that \( m\{\bar{D}\mu > \lambda\} = 0 \) for each \( \lambda > 0 \).

Therefore \( \bar{D}\mu = 0 \) a.e.\([m]\). \hfill \Box

**Example.** Let \( \mu \) be unit Dirac measure at \( x_0 \in \mathbb{R}^k \).

Then \( \mu \perp m \) and

\[
\lim_{r \to 0} \frac{\mu(B(x, r))}{m(B(x, r))} = \begin{cases} 
\infty & \text{if } x = x_0, \\
0 & \text{if } x \neq x_0.
\end{cases}
\]

Hence \( D\mu = 0 \) a.e.\([m]\).

On the other hand, \( D\mu = \infty \) a.e.\([\mu]\).

[The last statement is true for any positive Borel measure \( \mu \) such that \( \mu \perp m \) (see Theorem 7.15).]