Inverse Substitutions by Trigonometric (Hyperbolic) Functions. Reversing the Substitution Rule sometimes leads to simpler integrals: if \( x = g(t) \) for \( g \) invertible and differentiable, then \( dx = g'(t)dt \) and

\[
\int f(x) \, dx = \int f(g(t))g'(t) \, dt.
\]

This kind of inverse substitution can give \( f(g(t))g'(t) \) as simpler to integrate than \( f(x) \).

The choice of \( g \) as a trigonometric or hyperbolic trigonometric function can eliminate square roots from the integrand.

**Example 1.** Evaluate \( \int \frac{1}{x^2\sqrt{x^2-9}} \, dx \).

We can use the trigonometric identity \( \sec^2 \theta - 1 = \tan^2 \theta \) to choose an inverse substitution:

\[ x = 3 \sec \theta, \quad dx = 3 \sec \theta \tan \theta \, d\theta. \]

Why the factor of 3?

With this inverse substitution, the integral becomes

\[
\int \frac{1}{x^2\sqrt{x^2-9}} \, dx = \int \frac{3 \sec \theta \tan \theta}{9 \sec^2 \theta \sqrt{9 \sec^2 \theta - 9}} \, d\theta
= \int \frac{\sec \theta \tan \theta}{9 \sec^2 \theta \sqrt{\tan^2 \theta}} \, d\theta
= \int \frac{\sec \theta \tan \theta}{9 \sec^2 \theta |\tan \theta|} \, d\theta.
\]

Remember that \( \sqrt{y^2} = |y| \) in general, and that \( \sqrt{y^2} = y \) only if we known that \( y \geq 0 \).

So we ASSUME that \( \tan \theta \geq 0 \), so that \( |\tan \theta| = \tan \theta \), i.e., that \( 0 < \theta < \pi/2 \).

With the assumption, the integral becomes

\[
\int \frac{1}{x^2\sqrt{x^2-9}} \, dx = \frac{1}{9} \int \frac{1}{\sec \theta} \, d\theta
= \frac{1}{9} \int \cos \theta \, d\theta
= \frac{\sin \theta}{9} + C.
\]

We must express this indefinite integral in terms of the original variable \( x \), but how?

The function \( \sec \) in \( x = 3 \sec \theta \) is invertible on \((0, \pi/2)\), and so

\[ \theta = \sec^{-1}(x/3). \]
Thus the indefinite integral is
\[
\int \frac{1}{x^2\sqrt{x^2 - 9}} \, dx = \frac{\sin(\sec^{-1}(x/3))}{9} + C.
\]
The composition of sine with the inverse of secant is messy. Can it be simplified?
Yes, it can with the use of a right-angle triangle: one angle is \(\theta\) which lies between 0 and \(\pi/2\), and since \(\sec \theta = x/3\), the side of the triangle adjacent to the angle \(\theta\) has length 3 and the hypothenuse has length \(x\).
The Pythagorean Theorem then gives the length of the side opposite \(\theta\) as \(\sqrt{x^2 - 9}\), and so
\[
\sin(\sec^{-1}(x/3)) = \sin \theta = \frac{\sqrt{x^2 - 9}}{x}.
\]
The indefinite integral is then
\[
\int \frac{1}{x^2\sqrt{x^2 - 9}} \, dx = \frac{\sqrt{x^2 - 9}}{9x} + C.
\]
We can (and should) verify this (especially after all the work we did to get it):
\[
\frac{d}{dx} \frac{\sqrt{x^2 - 9}}{9x} = \frac{(1/2)(x^2 - 9)^{-1/2}(2x)(9x) - 9\sqrt{x^2 - 9}}{81x^2}
= \frac{x^2(x^2 - 9)^{-1/2} - \sqrt{x^2 - 9}}{9x^2} \cdot \frac{\sqrt{x^2 - 9}}{x^2 - 9}
= \frac{x^2 - (x^2 - 9)}{9x^2\sqrt{x^2 - 9}}
= \frac{1}{x^2\sqrt{x^2 - 9}}.
\]

**Example.** Evaluate \(\int x^3\sqrt{9 - x^2} \, dx\).

The trigonometric identity \(\sin^2 \theta + \cos^2 \theta = 1\) suggests the inverse substitution of
\[
x = 3 \sin \theta, \quad dx = 3 \cos \theta \, d\theta.
\]
For \(\theta\) in \((0, \pi/2)\), the indefinite integral becomes
\[
\int x^3\sqrt{9 - x^2} \, dx = \int 27 \sin^3 \theta \sqrt{9 - 9 \sin^2 \theta} (3 \cos \theta) \, d\theta
= 243 \int \sin^3 \theta \cos^2 \theta \, d\theta.
\]
With the power of sine being odd, we use the substitution \(u = \cos \theta, \, du = -\sin \theta d\theta\) to convert the integrand into a polynomial:
\[\int x^3 \sqrt{9 - x^2} \, dx = 243 \int (1 - \cos^2 \theta) \cos \theta \sin \theta \, d\theta\]
\[= -243 \int (1 - u^2)u^2 \, du\]
\[= -243 \int (u^2 - u^4) \, du\]
\[= -243 \left[ \frac{u^3}{3} - \frac{u^5}{5} \right] + C.\]

Since \( x = 3 \sin \theta \) and \( \theta \) is in \((0, \pi/2)\) where \( \sin \) is invertible, we get \( \theta = \sin^{-1}(x/3). \)

Since \( u = \cos \theta \), we get
\[\int x^3 \sqrt{9 - x^2} \, dx = -243 \left[ \frac{\cos^3(\sin^{-1}(x/3))}{3} - \frac{\cos^5(\sin^{-1}(x/3))}{5} \right] + C.\]

Again, by a right-angled triangle with \( \theta \) as one angle, \( x \) as the side opposite \( \theta \), and 3 as the hypotenuse, we get that
\[\cos(\sin^{-1}(x/3)) = \frac{\sqrt{9 - x^2}}{3}.\]

Thus the indefinite integral is
\[\int x^3 \sqrt{9 - x^2} \, dx = -243 \left[ \frac{(9 - x^2)^{3/2}}{81} - \frac{(9 - x^2)^{5/2}}{243 \times 5} \right] + C\]
\[= -3(9 - x^2)^{3/2} + \frac{(9 - x^2)^{5/2}}{5} + C.\]

We may think that this does not “look” right, but we can tell for sure by verification:
\[\frac{d}{dx} \left[ -3(9 - x^2)^{3/2} + \frac{(9 - x^2)^{5/2}}{5} + C \right] = -\frac{9}{2} \sqrt{9 - x^2}(-2x) + \frac{1}{2}(9 - x^2)^{3/2}(-2x)\]
\[= 9x \sqrt{9 - x^2} - x(9 - x^2)^{3/2}\]
\[= x \sqrt{9 - x^2}(9 - (9 - x^2))\]
\[= x^3 \sqrt{9 - x^2}.\]

Evaluation of Definite Integrals by Inverse Substitutions. When we deal with definite integrals, we do not have to undo all the changes we made along the way.

**Example 3.** Evaluate \( \int_{0}^{1} \sqrt{x^2 + 1} \, dx. \)

The trigonometric identity \( 1 + \tan^2 \theta = \sec^2 \theta \) suggest the inverse substitution
\[x = \tan \theta, \quad dx = \sec^2 \, d\theta.\]
Here the limits of integration become 0 = \tan u or \ u = 0, and 1 = \tan u or \ u = \pi/4, so that the definite integral becomes
\[
\int_0^1 \sqrt{x^2 + 1} \, dx = \int_0^{\pi/4} \sqrt{\tan^2 \theta + 1} \ \sec^2 \theta \, d\theta
\]
\[
= \int_0^{\pi/4} \sec^3 \theta \, d\theta.
\]

With the power of secant being odd, there is no trigonometric identity that will help us here.

Instead we opt for an integration by parts approach: with

\[u = \sec \theta, \ \ dv = \sec^2 \theta \, d\theta, \ \ du = \sec \theta \tan \theta \, d\theta, \ \ v = \tan \theta,\]

the definite integral becomes
\[
\int_0^{\pi/4} \sec^3 \theta \, d\theta = \sec \theta \tan \theta \bigg|_0^{\pi/4} - \int_0^{\pi/4} \sec \theta \tan^2 \theta \, d\theta
\]
\[
= \frac{2}{\sqrt{2}} - \int_0^{\pi/4} \sec \theta (\sec^2 \theta - 1) \, d\theta
\]
\[
= \frac{2}{\sqrt{2}} - \int_0^{\pi/4} \sec^3 \theta \, d\theta + \int_0^{\pi/4} \sec \theta \, d\theta.
\]

Combining the two integrals involving the cube of secant gives
\[
\int_0^{\pi/4} \sec^3 \theta \, d\theta = \frac{1}{\sqrt{2}} + \frac{1}{2} \int_0^{\pi/4} \sec \theta \, d\theta
\]
\[
= \frac{1}{\sqrt{2}} + \frac{1}{2} \left[ \ln |\sec \theta + \tan \theta| \right]_0^{\pi/4}
\]
\[
= \frac{1}{\sqrt{2}} + \frac{1}{2} \left[ \ln \left| \frac{2}{\sqrt{2}} + 1 \right| - \ln |1 + 0| \right]
\]
\[
= \frac{1}{\sqrt{2}} + \frac{1}{2} \ln(1 + \sqrt{2}).
\]

Along the way, we learned that
\[
\int \sec^3 \theta \, d\theta = \frac{\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|}{2} + C,
\]

from using integration by parts and
\[
\int \sec \theta \, d\theta = \ln |\sec \theta + \tan \theta| + C.
\]