Polynomial Division. Integration of a rational function requires techniques from algebra to convert the rational function into a form that is easily integrated.

If the degree of the polynomial $P(x)$ in the numerator is the same or bigger than the degree of the polynomial $Q(x)$ in the denominator, the first bit of algebra to be done is polynomial division:

$$f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

where $S(x)$ and $R(x)$ are polynomial with the degree of $R(x)$ being strictly smaller than that of $Q(x)$.

There is no problem in integrating $S(x)$ since it is a polynomial.

**Example 1.** Find $S(x)$ and $R(x)$ by applying polynomial division to

$$f(x) = \frac{P(x)}{Q(x)} = \frac{x^2}{x + 4}.$$ 

The answer is

$$\frac{x^2}{x + 4} = x - 4 + \frac{16}{x + 4}.$$ 

This can (and should) be checked:

$$x - 4 + \frac{16}{x + 4} = \frac{(x - 4)(x + 4) + 16}{x + 4} = \frac{x^2 - 16 + 16}{x + 4} = \frac{x^2}{x + 4}.\sqrt{\text{.}}$$

Thus $S(x) = x - 4$ and $R(x) = 16$.

**Example 2.** Use polynomial division to simplify

$$\frac{x^3}{x^2 + 2x + 1}.$$ 

The simplification obtained is

$$\frac{x^3}{x^2 + 2x + 1} = x - 2 + \frac{3x + 2}{x^2 + 2x + 1}.$$ 

Partial Fractions with Distinct Linear Factors. Now we suppose that in the rational function $R(x)/Q(x)$, with the degree of $R$ strictly less than that of $Q(x)$, that $Q(x)$ is a product of distinct linear factors:

$$Q(x) = (a_1x + b_1)(a_2x + b_2) \cdots (a_kx + b_k),$$

where $k$ is the degree of $Q$. 
In this case there is a partial fraction theorem that states that

\[ \frac{R(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_k}{a_kx + b_k}. \]

The point of this is that each of the terms of the partial fraction decomposition (the terms on the right side) is easily integrated.

**Example 1 Continued.** Evaluate \( \int \frac{x^2}{x + 4} \, dx \).

We saw before that by polynomial division the integrand simplifies where the rational part has one linear factor in the denominator:

\[
\int \frac{x^2}{x + 4} \, dx = \int \left( x - 4 + \frac{16}{x + 4} \right) \, dx = \frac{x^2}{2} - 4x + 16 \ln |x + 4| + C.
\]

**Example 3.** Evaluate \( \int \frac{x - 2}{x^2 - 9x - 10} \, dx \).

The denominator factors into linear terms:

\[ x^2 - 9x - 10 = (x - 10)(x + 1). \]

The partial fraction decomposition of the integrand is then

\[ \frac{x - 2}{(x - 10)(x + 1)} = \frac{A}{x - 10} + \frac{B}{x + 1}. \]

The values of the constants \( A \) and \( B \) are found by equating the numerators (after finding the common denominator):

\[ x - 2 = A(x + 1) + B(x - 10). \]

Multiplying this out gives two linear equations in two unknowns, which can be solved for \( A \) and \( B \).

But with distinct linear factors there is a short-cut: the partial decomposition works for all values of \( x \), so it works for some values of \( x \), say \( x = -1 \) and \( x = 10 \).

Why are these values of \( x \) singled out? Because each eliminates one of the unknown constants from the equation:

\[ x = -1 : \ ( -1 ) - 2 = A(-1 + 1) + B(-1 - 10) = -11B \quad \Rightarrow \quad B = \frac{3}{11}, \]

\[ x = 10 : \ 10 - 2 = A(10 + 1) + B(10 - 10) = 11A \quad \Rightarrow \quad A = \frac{8}{11}. \]

The partial fraction decomposition is

\[ \frac{x - 2}{x^2 - 9x - 10} = \frac{8/11}{x - 10} + \frac{3/11}{x + 1}. \]
You can (and should) check this:

\[
\frac{8/11}{x-10} + \frac{3/11}{x+1} = \frac{(8/11)(x+1) + (3/11)(x-10)}{(x-10)(x+1)} = \frac{x-22/11}{x^2-9x-10} = \frac{x-2}{x^2-9x-10}.
\]

The indefinite integral is

\[
\int \frac{x-2}{x^2-9x-10} \, dx = \int \left( \frac{8/11}{x-10} + \frac{3/11}{x+1} \right) \, dx = \frac{8 \ln |x-10|}{11} + \frac{3 \ln |x+1|}{11} + C.
\]

Partial Fractions with Linear Factors, Some Repeated. When a linear factor is repeated, the form of the partial fraction decomposition changes.

If in \( R(x)/Q(x) \), the polynomial \( Q(x) \) factors, say for \( n \geq 2 \), as

\[
(a_1x + b_1)^n(a_2x + b_2) \cdots (a_kx + b_k)
\]

then the partial fraction decomposition is

\[
\frac{R(x)}{Q(x)} = \frac{A_{11}}{a_1x + b_1} + \frac{A_{12}}{(a_1x + b_1)^2} + \cdots + \frac{A_{in}}{(a_1x + b_1)^n} + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_k}{a_kx + b_k}.
\]

If more than one linear factor is repeated, each repeated factor corresponds to the same pattern given above for one repeated linear factor.

**Example 2 Continued.** Evaluate \( \int \frac{x^3}{x^2 + 2x + 1} \, dx \).

We saw before that by polynomial division, the integrand becomes

\[
\frac{x^3}{x^2 + 2x + 1} = x - 2 + \frac{3x + 2}{x^2 + 2x + 1}.
\]

The denominator factors into repeated linear factors:

\[
x^2 + 2x + 1 = (x+1)^2.
\]

The partial fraction decomposition is

\[
\frac{3x + 2}{x^2 + 2x + 1} = \frac{A}{x+1} + \frac{B}{(x+1)^2}.
\]

Equating the numerators of both sides (after finding the common denominator) gives

\[
3x + 2 = A(x+1) + B = Ax + A + B.
\]

From this \( A \) must be 3 and so \( 2 = A + B = 3 + B \) requires that \( B = -1 \).
The partial fraction decomposition is
\[
\frac{3x + 2}{x^2 + 2x + 1} = \frac{3}{x + 1} + \frac{-1}{(x + 1)^2}.
\]
You can (and should) check this:
\[
\frac{3}{x + 1} + \frac{-1}{(x + 1)^2} = \frac{3(x + 1) - 1}{(x + 1)^2} = \frac{3x + 2}{x^2 + 2x + 1}.
\]
Now the indefinite integral can be evaluated:
\[
\int \frac{x^3}{x^2 + 2x + 1} \, dx = \int \left(x - 2 + \frac{3x + 2}{x^2 + 2x + 1}\right) \, dx
\]
\[
= \int (x - 2) \, dx + \int \left(\frac{3}{x + 1} + \frac{-1}{(x + 1)^2}\right) \, dx
\]
\[
= \frac{x^2}{2} - 2x + 3 \ln |x + 1| + \frac{1}{x + 1} + C.
\]