Partial Fractions with Distinct Irreducible Quadratic Factors. Suppose for a rational function \( R(x)/Q(x) \), with the degree of \( R \) strictly smaller than the degree of \( Q \). Suppose further that the factorization of \( Q \) has a quadratic term \( ax^2 + bx + c \), where the discriminant \( b^2 - 4ac < 0 \), i.e., it has complex roots, or is irreducible.

The partial fraction theory states that the decomposition for an irreducible quadratic factor has the form

\[ \frac{Ax + B}{ax^2 + bx + c} \]

**Example 1.** Evaluate \( \int \frac{x + 4}{x^2 + 2x + 5} \, dx \).

The discriminant of the quadratic denominator is \( 4 - 20 = -16 < 0 \), and so it is irreducible.

The rational function is already in the partial fraction form.

So what do we do now????

We complete the square of the denominator first:

\[ x^2 + 2x + 5 = x^2 + 2x + 1 - 1 + 5 = (x + 1)^2 + 4. \]

We then split-up the indefinite integral:

\[
\int \frac{x + 4}{x^2 + 2x + 5} \, dx = \int \frac{x + 4}{(x + 1)^2 + 4} \, dx = \int \frac{x + 1}{(x + 1)^2 + 4} \, dx + \int \frac{3}{(x + 1)^2 + 4} \, dx.
\]

Now we apply Inverse Substitution to each indefinite integral: we use

\[ 2u = x + 1, \quad 2du = dx \]

for both, which then become

\[
\int \frac{x + 4}{x^2 + 2x + 5} \, dx = \int \frac{4u}{(2u)^2 + 4} \, du + \int \frac{6}{(2u)^2 + 4} \, du = \frac{1}{2} \int \frac{2u}{u^2 + 1} \, du + \frac{3}{2} \int \frac{1}{u^2 + 1} \, du.
\]

The first integral is a natural logarithm, and the second integral is an arctan:

\[
\int \frac{x + 4}{x^2 + 2x + 5} \, dx = \frac{\ln|u^2 + 1|}{2} + \frac{3}{2} \arctan(u) + C.
\]
Undoing the reverse substitution $2u = x + 1$ gives

$$
\int \frac{x + 4}{x^2 + 2x + 5} \, dx = \frac{1}{2} \ln \left( \frac{(x + 1)^2}{2} + 1 \right) + \frac{3}{2} \arctan \left( \frac{x + 1}{2} \right) + C
$$

$$
= \frac{1}{2} \ln \left( \frac{x^2 + 2x + 1}{4} + 1 \right) + \frac{3}{2} \arctan \left( \frac{x + 1}{2} \right) + C
$$

$$
= \frac{1}{2} \ln \left( \frac{x^2 + 2x + 1 + 4}{4} \right) + \frac{3}{2} \arctan \left( \frac{x + 1}{2} \right) + C
$$

$$
= \frac{1}{2} \ln (x^2 + 2x + 5) - \frac{\ln 4}{2} + \frac{3}{2} \arctan \left( \frac{x + 1}{2} \right) + C
$$

$$
= \frac{1}{2} \ln (x^2 + 2x + 5) + \frac{3}{2} \arctan \left( \frac{x + 1}{2} \right) + C.
$$

The constant $(\ln 4)/2$ has been absorbed into the arbitrary constant $C$.

**Example 2.** Evaluate $\int \frac{1}{(x^2 + 1)(x^2 + 4)} \, dx$.

Because the factors in the denominator are irreducible and distinct, the partial fraction decomposition is

$$\frac{1}{(x^2 + 1)(x^2 + 4)} = \frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{x^2 + 4}.$$  

The constants $A, B, C, D$ must satisfy

$$1 = (Ax + B)(x^2 + 4) + (Cx + D)(x^2 + 1).$$

Unfortunately, there are no real zeroes to use here to quickly find $A, B, C, D$.

Multiplying the products gives

$$1 = Ax^3 + Bx^2 + 4Ax + 4B + Cx^3 + Dx^2 + Cx + D.$$  

For these two polynomials to be equal requires that the coefficients of like degree be the same:

- $x^3$: $A + C = 0$,
- $x^2$: $B + D = 0$,
- $x$: $4A + C = 0$,
- $1$: $4B + D = 1$.

Solving the first and third equations in $A$ and $C$ simultaneously gives $A = 0$ and $C = 0$.

Solving the second and fourth equations in $B$ and $D$ simultaneously gives

$$1 = 4B + D = 4B - B = 3B \implies B = 1/3 \text{ and } D = -1/3.$$  

So the partial fraction decomposition is

$$\frac{1}{(x^2 + 1)(x^2 + 4)} = \frac{1/3}{x^2 + 1} + \frac{-1/3}{x^2 + 4}.$$
We can make use of the integral formula
\[
\int \frac{1}{x^2 + a^2} \, dx = \frac{1}{a} \arctan \left( \frac{x}{a} \right) + C
\]
to carry out the integration:
\[
\int \frac{1}{(x^2 + 1)(x^2 + 4)} \, dx = \frac{1}{3} \arctan(x) - \frac{1}{6} \arctan \left( \frac{x}{2} \right) + C.
\]

Partial Fractions with Repeated Irreducible Quadratic Factors. When there are repeated irreducible quadratic factors in the denominator of a rational function, the partial fraction form for each repeated irreducible quadratic factor is
\[
\frac{R(x)}{(ax^2 + bx + c)^k} = \frac{A_1 x + B_1}{ax^2 + bx + c} + \frac{A_2 x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_k x + B_k}{(ax^2 + bx + c)^k}.
\]

Example 3. Evaluate \( \int \frac{dx}{x(x^2 + 4)^2} \).

The partial fraction decomposition for the integrand is
\[
\frac{1}{x(x^2 + 4)^2} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4} + \frac{Dx + E}{(x^2 + 4)^2}.
\]
The five constants here satisfy
\[
1 = A(x^2 + 4)^2 + (Bx + C)(x)(x^2 + 4) + (Dx + E)(x)
\]
\[
= A(x^4 + 8x^2 + 16) + (Bx^2 + Cx)(x^2 + 4) + Dx^2 + Ex
\]
\[
= Ax^4 + 8Ax^2 + 16A + Bx^4 + Cx^3 + 4Bx^2 + 4Cx + Dx^2 + Ex.
\]
Equating coefficients of like power gives
\[
A = \frac{1}{16}, \quad 4C + E = 0, \quad 8A + 4B + D = 0, \quad C = 0, \quad A + B = 0.
\]
From these we get
\[
E = 0, \quad B = -\frac{1}{16}, \quad D = -8A - 4B = -\frac{1}{2} + \frac{1}{4} = -\frac{1}{4}.
\]
The partial fraction decomposition is
\[
\frac{1}{(x^2 + 1)(x^2 + 4)} = \frac{1/16}{x} + \frac{-x/16}{x^2 + 4} - \frac{x/4}{(x^2 + 4)^2}.
\]
Now we can carry out the integration:
\[
\int \frac{1}{(x^2 + 1)(x^2 + 4)} \, dx = \frac{\ln |x|}{16} - \frac{\ln(x^2 + 4)}{32} + \frac{1}{8} \left( \frac{1}{x^2 + 4} \right) + C.
\]
Rationalizing Substitutions. Sometimes non-rational functions can be transformed into rational functions by substitution.

In particular, when there are non-rational functions like $x^{1/n}$ in the integrand, the inverse substitution $x = u^n$ might rationalize the integrand.

**Example 4.** Evaluate $\int_0^1 \frac{1}{1 + x^{1/3}} \, dx$.

We try the substitution $x = u^3, \quad du = 3u^2 \, dx$.

This gives

$$\int \frac{1}{1 + x^{1/3}} \, dx = \int \frac{3u^2}{1 + u} \, du.$$

We now have a rational function, and proceed to integrate it.