Math 113 Lecture #20
§8.5: Probability

The lifetime of a randomly chosen battery of a given type is a continuous random variable $X$: the lifetime $X$ is a real number that ranges over an interval of real numbers.

The probability that the lifetime of such a battery is at least 1000 hours but not more than 2000 hours is denoted by

$$P(1000 \leq X \leq 2000).$$

The probability that the lifetime of a battery is at least 500 hours is denoted by

$$P(X \geq 500).$$

The probability that the lifetime of a battery is no more than 1500 hours is denoted by

$$P(X \leq 1500).$$

Each of these probabilities is a number between 0 and 1.

How do we associate a number between 0 and 1 to a given probability?

Each continuous random variable $X$ has a nonnegative probability density function $f(x)$ associated with it.

The probability that $X$ lies between two values $a$ and $b$ is given by integration:

$$P(a \leq X \leq b) = \int_a^b f(x) \, dx.$$

The probability that a continuous random variable is a real number is 1, which means that

$$P(-\infty < X < \infty) = \int_{-\infty}^{\infty} f(x) \, dx = 1.$$

Example. Is the nonnegative function $f(x) = xe^{-x}$ for $x \geq 0$ and $f(x) = 0$ for $x < 0$, a probability density function?

Here is the graph of $f(x)$. 

[Graph of $f(x)$]
To determine if it is or is not, we compute the value of the hopefully convergent improper integral to see if it is 1:

\[
\int_{-\infty}^{\infty} f(x) = \int_{0}^{\infty} xe^{-x}dx \\
= \lim_{A \to \infty} \int_{0}^{A} xe^{-x}dx \\
= \lim_{A \to \infty} \left[ -xe^{-x} - e^{-x} \right]_{0}^{A} \\
= \lim_{A \to \infty} \left[ -Ae^{-A} - e^{-A} + 1 \right] = 1.
\]

Thus \( f(x) \) is a probability density function, and we can use it to compute probabilities.

The probability that a continuous random variable \( X \) with probability density function \( X \) has a value between 1 and 2 is

\[
P(1 \leq X \leq 2) = \int_{1}^{2} xe^{-x}dx \\
= \left[ -xe^{-x} - e^{-x} \right]_{1}^{2} \\
= -2e^{-2} - e^{-1} + e^{-1} \\
\approx 0.3297530327
\]

There is an approximately 33% change that the value of \( X \) is between 1 and 2.

Suppose you are waiting for a company to answer your phone call.

How long can you expect to wait?

Let \( f(t) \) be the probability density function where \( t \) is time measured in minutes.

For a sample of \( N \) people who have called, we can assume that none will wait more than 60 minutes.

On a small subinterval \([t_{i-1}, t_i]\) of length \( \Delta t \) of \([0, 60]\) the probability that someone’s call gets answers is approximately \( f(\bar{t}_i)\Delta t \) for some \( \bar{t}_i \in [t_{i-1}, t_i] \).

Of the \( N \) people who have called, we expect that during \([t_{i-1}, t_i]\) the number of calls that will be answered is approximately \( N f(\bar{t}_i)\Delta t \), and the time they each waited is about \( \bar{t}_i \).

So the total of the waiting times over \([0, 60]\) is

\[
\sum_{i=1}^{n} N\bar{t}_i f(\bar{t}_i)\Delta t.
\]

Then the average waiting time is

\[
\frac{1}{N} \sum_{i=1}^{n} N\bar{t}_i f(\bar{t}_i)\Delta t = \sum_{i=1}^{n} \bar{t}_i f(\bar{t}_i)\Delta t.
\]
Recognizing this as a Riemann sum for the function $tf(t)$, we obtain in the limit that the average waiting time or mean is

$$\int_{0}^{60} tf(t) \, dt.$$ 

Passing to an improper integral we define the **mean** of a probability density function $f(t)$ to be

$$\mu = \int_{-\infty}^{\infty} tf(t) \, dt.$$ 

Another measure for a probability density function is its **median**, the number $m$ that satisfies

$$\int_{m}^{\infty} f(x) \, dx = \frac{1}{2}.$$ 

Unlike the mean, finding the median may involve a numerical approximation for the value of $m$.

**Example.** An exponentially decreasing probability density function is commonly used to model wait times or equipment failure times.

For each $c > 0$ an exponentially decreasing probability density function is

$$f(x) = \begin{cases} 
    ce^{-cx} & \text{if } x \geq 0, \\
    0 & \text{if } x < 0.
\end{cases}$$

Here is the graph of $f(x)$ for $c = 2$.

For arbitrary $c > 0$, the mean of $f(x)$ is

$$\mu = \int_{-\infty}^{\infty} xf(x) \, dx = \int_{0}^{\infty} cxe^{-cx} \, dx$$

$$= \lim_{A \to \infty} \int_{0}^{A} cxe^{-cx} \, dx = \lim_{A \to \infty} \left[ -xe^{-cx} - \frac{e^{-cx}}{c} \right]_{0}^{A}$$

$$= \lim_{A \to \infty} \left[ -Ae^{-cA} - \frac{e^{-cA}}{c} + \frac{1}{c} \right] = \frac{1}{c}.$$
Thus the average waiting time is $1/c$.

The median $m$ of $f(x)$ satisfies
\[
\frac{1}{2} = \int_m^\infty ce^{-cx} \, dx = \lim_{A \to \infty} \left[ -e^{-cx} \right]_m^A = \lim_{A \to \infty} \left[ -e^{-cA} + e^{-cm} \right] = e^{-cm}.
\]

We can solve this for
\[
m = \frac{\ln 2}{c}.
\]

For $c = 2$ the median is $m \approx 0.346$ which is different than the mean of 0.5.

**Normal Distributions.** Many continuous random variables $X$, e.g. scores on exams, have a normal probability density distribution which is given by
\[
\frac{1}{\sigma \sqrt{2\pi}} \exp \left( -\frac{(x - \mu)^2}{2\sigma^2} \right)
\]
where $\mu$ is the mean of $f(x)$ and $\sigma$ is the standard deviation that measures how spread out the values of $X$.

The smaller the value of $\sigma$ the more concentrated the probability density is near the mean, and the larger the value of $\sigma$ the less concentrated the probability density is near the mean.

Here is a “curve” for an exam with a mean of 70 and a standard deviation of 12.

Computing probability with a normal distribution requires numerical integration to approximate the integrals.

For a normal distribution with mean 70 and standard deviation of 12, the probability that a student gets an exam score of between 70 and 90 is
\[
\int_{70}^{90} \frac{1}{12\sqrt{2\pi}} \exp \left( -\frac{(x - 70)^2}{2(12^2)} \right) \, dx \approx 0.4522096478.
\]
For the same normal curve, the probability that a student gets an exam score of at least 80 is

$$\int_{80}^{\infty} \frac{1}{12\sqrt{2\pi}} \exp \left( -\frac{(x - 70)^2}{2(12^2)} \right) dx \approx \int_{80}^{120} \frac{1}{12\sqrt{2\pi}} \exp \left( -\frac{(x - 70)^2}{2(12^2)} \right) dx \approx 0.2023129267.$$

Why could we replace $\infty$ with 120? Because for $x > 120$ there isn’t much probability, or the area under the curve over $x > 120$ is very very small, and can safely be ignored.