Name:______________________________
Student ID(see bubble sheet):______________
Section:______________________________
Instructor:

Math 313 (Linear Algebra)
Exam 3 Practice
Mar 26,27,28

Instructions:

• For questions which require a written answer, show all your work. Full credit will be given only if the necessary work is shown justifying your answer.
• Simplify your answers.
• Scientific calculators are allowed.
• Should you have need for more space than is allocated to answer a question, use the back of the page the problem is on and indicate this fact.
• Please do not talk about the test with other students until after the last day to take the exam.
Part I: Multiple Choice Questions: Mark all answers which are correct for each question. (4 points each.)

1. On \(\mathbb{P}_2\) the linear transformation \(T(p)(x) := x^2 p''(x)\) is given. Which of the following statements are true?
   
   a) \(T\) is invertible. 
   b) \(T\) has a kernel of dimension 2. 
   c) \(T\) is onto. 
   d) The matrix of \(T\) w.r.t. the basis \(B := \{1, t, t^2\}\) assumes the form 
      \[
      \begin{pmatrix}
      0 & 0 & 0 \\
      0 & 0 & 0 \\
      1 & 0 & 0 \\
      \end{pmatrix}
      \]
   
   e) The matrix of \(T\) w.r.t. the basis \(B := \{1, t, t^2\}\) assumes the form 
      \[
      \begin{pmatrix}
      0 & 0 & 0 \\
      0 & 0 & 0 \\
      0 & 0 & 1 \\
      \end{pmatrix}
      \]
   f) The matrix of \(T\) w.r.t. the basis \(B := \{1, t, t^2\}\) cannot be a square matrix.

2. Let \(W = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}\). Which of the following sets of vectors span the subspace \(W^\perp\)?
   
   Select all that apply:
   
   a) \(\left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \right\}\) 
   b) \(\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}\) 
   c) \(\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}\) 
   d) \(\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}\) 
   e) \(\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}\) 
   f) \(\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix} \right\}\)

3. Let \(A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\). Then, the matrix \(P\) that diagonalizes \(A\) is given by
   
   a) \(\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}\) 
   b) \(\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}\) 
   c) \(\begin{bmatrix} 2 & 1 \\ 2 & -1 \end{bmatrix}\) 
   d) \(\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\) 
   e) \(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\) 
   f) \(\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}\)
Part II: Fill in the blank with the best possible answer. (4 points each.)

4. (10 points)

(a) A matrix $A$ is called symmetric if $A^T = A$. Let $M$ denote the vector space of all symmetric $2 \times 2$ matrices. Then $\dim M = \underline{\phantom{0}}$.

(b) Fill in the matrix so that it has rank 1.

\[
\begin{bmatrix}
1 & 2 \\
2 & -4 \\
& -6 \\
\end{bmatrix}
\]

(c) Let $\{u_1, \ldots, u_p\}$ be an orthogonal basis for a subspace $W$ of $\mathbb{R}^n$. For each $y$ in $W$, the weights of the linear combination $y = c_1u_1 + \cdots + c_pu_p$ are given by

\[c_j = \underline{\phantom{0}} \quad \text{for } j = 1, \ldots, p.\]

(d) Let $y = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$, and $W = \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$. The distance from $y$ to $W$ is $\underline{\phantom{0}}$.

(e) If $A$ is an $n \times n$ matrix, $\mathbf{x}$ is a nonzero vector, and $A\mathbf{x} = \lambda \mathbf{x}$ for some scalar $\lambda$, then $\underline{\phantom{0}}$.

(f) Assume $A$ is an $n \times n$ matrix that has distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_r$ with corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$. What can be said of the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$? $\underline{\phantom{0}}$.
Part III: Justify your answer and show all work for full credit.

5. For a real 2x2 matrix $A$ prove or disprove by giving a counterexample that with every complex eigenvector $u + iv$ also its conjugate, $u - iv$, is an eigenvector of $A$.

6. Let $T : V \to W$ be a linear transformation that is one-to-one. Prove that $\dim V \leq \dim W$.

   Hint: take a basis $\{b_1, \ldots, b_n\}$ of $V$, and consider the set $\{T(b_1), \ldots, T(b_n)\}$ in $W$.

7. Can

   $$F = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

   be diagonalized? If it can, find the diagonalizing matrix. Otherwise, show why it cannot.

8. Prove that for any $m \times n$ matrix $A$, we have $\dim(\text{Nul} A)^\perp = \dim(\text{Nul} A^T)^\perp$.

9. Let $W = \text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ -5 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$. Find a basis for $W^\perp$.

10. Let $A$ be a $15 \times 20$ matrix. For a particular vector $b$, the system $Ax = b$ has 6 free variables. Can we guarantee that $Ax = b$ always has a solution for any vector $b$? Justify your conclusion.

11. Prove that if $A$ and $B$ are similar matrices, then $A$ and $B$ have the same eigenvalues.

12. Let $W = \left\{ \begin{bmatrix} a \\ b \\ a \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$. Use an orthogonal basis for $W$ and the projection formula to find the closest point in $W$ to $y = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$. Does your answer makes sense? Explain.

13. For the matrix $A$ given by

   $$A = \begin{pmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{pmatrix}$$

   (a) Find its eigenvalues and eigenvectors

   (b) Check that the eigenvalues and eigenvectors of $A$ found in part (a) are correct by verifying that $Av = \lambda v$ for every eigenvalue $\lambda$ and its corresponding eigenvector $v$.

END OF EXAM