Math 313 Lecture #6  
§1.7: Linear Independence

We are going to use the homogeneous system $A\vec{x} = \vec{0}$ to analyze a critically important concept for vectors in $\mathbb{R}^n$.

Consider in $\mathbb{R}^3$ the three vectors

\[
\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.
\]

Notice that $\vec{v}_3 = \vec{v}_1 - \vec{v}_2$, and so one vector is a linear combination of the others.

Rewritten we have the linear combination $\vec{v}_1 - \vec{v}_2 - \vec{v}_3 = \vec{0}$, or as a matrix equation,

\[
\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.
\]

**Definitions.** Vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ in $\mathbb{R}^n$ are said to be **linearly dependent** if there are weights $c_1, c_2, \ldots, c_p$, not all zero, such that

\[
c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_p \vec{v}_p = \vec{0}.
\]

This equation is called a **linear dependence relation** for linearly dependent vectors.

Vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ in $\mathbb{R}^n$ are said to be **linearly independent** if the only solution of

\[
x_1 \vec{v}_1 + x_2 \vec{v}_2 + \cdots + x_p \vec{v}_p = \vec{0}
\]

is the trivial one, i.e., $x_1 = 0, x_2 = 0, \ldots, x_p = 0$.

Since we can switch between vector equations and matrix equations, the linear dependence or linear independence of vectors is connected with a homogeneous system.

For $A = [\vec{a}_1 \quad \vec{a}_2 \quad \cdots \quad \vec{a}_p]$ we have that $A\vec{x} = \vec{0}$ has the same solution set as that of

\[
x_1 \vec{a}_1 + x_2 \vec{a}_2 + \cdots + x_p \vec{a}_p = \vec{0}.
\]

So the columns of $A$ are linearly independent if and only if the homogeneous system $A\vec{x} = \vec{0}$ has only the trivial solution.

**Example** Are the columns of

\[
A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 4 \\ 6 & -2 & 23 \end{bmatrix}
\]

linearly independent?
We answer the question by row reducing the matrix:

\[
\begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 4 \\
6 & -2 & 23
\end{bmatrix}
\]

\[ R_1 \leftrightarrow R_2 \]

\[ \sim \begin{bmatrix}
1 & 0 & 4 \\
0 & 1 & 0 \\
6 & -2 & 23
\end{bmatrix}
\]

\[ R_3 - 6R_1 \rightarrow R_3 \]

\[ \sim \begin{bmatrix}
1 & 0 & 4 \\
0 & 1 & 0 \\
0 & -2 & -1
\end{bmatrix}
\]

\[ R_3 + 2R_2 \rightarrow R_3 \]

\[ \sim \begin{bmatrix}
1 & 0 & 4 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{bmatrix}
\].

This says that there are no free variables, and so \( A \vec{x} = \vec{0} \) has only the trivial solution, meaning that the columns of \( A \) are linearly independent.

A single vector \( \vec{v}_1 \) in \( \mathbb{R}^n \) is linearly independent if and only if \( \vec{v}_1 \neq \vec{0} \).

Why? Because \( x_1 \vec{v}_1 = \vec{0} \) implies that \( x_1 = 0 \) when \( \vec{v} \neq \vec{0} \).

Two vectors \( \vec{v}_1 \) and \( \vec{v}_2 \) in \( \mathbb{R}^n \) are linearly independent if and only if neither is a scalar multiple of the other.

Why? Because if \( \vec{v}_1 \) and \( \vec{v}_2 \) are linearly dependent, then \( c_1 \vec{v}_1 + c_2 \vec{v}_2 = \vec{0} \), without loss of generality (or WLOG for short) say \( c_1 \neq 0 \) gives

\[ \vec{v}_1 = -\frac{c_2}{c_1} \vec{v}_2, \]

and so one vector is a scalar multiple of the other.

On the other hand, if one of \( \vec{v}_1 \) and \( \vec{v}_2 \) is a scalar multiple of the other, say WLOG \( \vec{v}_2 = d\vec{v}_1 \), then

\[ -d\vec{v}_1 + \vec{v}_2 = \vec{0} \]

where the weight of \( \vec{v}_2 \) is not zero, and so \( \vec{v}_1 \) and \( \vec{v}_2 \) are linear dependent.

For three or more vectors, we can similarly understand when they are linearly dependent.

**Theorem 7.** Vectors \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p \) in \( \mathbb{R}^n \) are linearly dependent if and only if one of the \( p \) vectors is a linear combination of the other \( p - 1 \) vectors.

**Proof.** WLOG, suppose that \( \vec{v}_p \) is a linear combination of the other \( p - 1 \) vectors: there are weights \( \alpha_1, \alpha_2, \ldots, \alpha_{p-1} \) such that

\[ \vec{v}_p = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \cdots + \alpha_{p-1} \vec{v}_{p-1}. \]

Moving \( \vec{v}_p \) to the other side gives

\[ \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \cdots + \alpha_{p-1} \vec{v}_{p-1} - \vec{v}_p = \vec{0}. \]
Letting \( c_i = \alpha_i \) for \( i = 1, \ldots, n - 1 \), and \( c_p = -1 \) gives

\[
c_1 \vv_1 + c_2 \vv_2 + \cdots + c_n \vv_p = \vec{0}
\]

where not all of the \( c_i \)'s are zero, and thus the vectors \( \vv_1, \vv_2, \ldots, \vv_p \) are linearly dependent.

On the other hand, if the vectors \( \vv_1, \vv_2, \ldots, \vv_p \) are linearly dependent, then at least one of the \( c_i \)'s in

\[
c_1 \vv_1 + c_2 \vv_2 + \cdots + c_p \vv_p = \vec{0},
\]

can be chosen to be nonzero.

If WLOG we say it is \( c_p \), then solving for \( \vv_p \) gives

\[
\vv_p = -\frac{c_1}{c_p} \vv_1 - \frac{c_2}{c_p} \vv_2 - \cdots - \frac{c_{p-1}}{c_p} \vv_{p-1}.
\]

Thus \( \vv_p \) is a linear combination of the other \( p - 1 \) vectors. \( \square \)

The contrapositive of Theorem 7 gives a characterization of linear independence: vectors \( \vv_1, \vv_2, \ldots, \vv_p \) in \( \mathbb{R}^n \) are linearly independent if and only if none of the \( p \) vectors is a linear combination of the other \( p - 1 \) vectors.

We explore some of the connections between the subset spanned by vectors and linear independence of vectors.

**Example.** Suppose \( \vu \) and \( \vv \) are linearly independent vectors in \( \mathbb{R}^3 \).

Could either \( \vu \) or \( \vv \) be the zero vector? No, because if say \( \vu = \vec{0} \), then the vectors \( \vec{0} \) and \( \vv \) would be linearly dependent because \( 1 \vec{0} + 0 \vv = \vec{0} \).

A similar argument holds if \( \vv = \vec{0} \).

We know that \( \text{Span}(\vu, \vv) \) is a plane through the origin in \( \mathbb{R}^3 \).

For a vector \( \vw \) in \( \mathbb{R}^3 \), what can we say about the linear dependence or linearly independence of the vectors \( \vu \), \( \vv \), and \( \vw \)?

If \( \vw \) belongs to \( \text{Span}(\vu, \vv) \), then \( \vw \) is a linear combination of \( \vu \) and \( \vv \), and so by Theorem 7, the vectors \( \vu \), \( \vv \), and \( \vw \) are linearly dependent.

On the other hand, if the vectors \( \vu \), \( \vv \), and \( \vw \) are linearly dependent, then by Theorem 7, one of the three vectors is a linear combination of the other two.

There are two cases to consider.

Case 1. If \( \vw \) is a linear combination of \( \vu \) and \( \vv \), then \( \vw \) is in \( \text{Span}(\vu, \vv) \).

Case 2. WLOG, suppose that the nonzero \( \vu \) is a linear combination of \( \vv \) and \( \vw \).

Then there are weights \( c_1 \) and \( c_2 \) such that \( \vu = c_1 \vv + c_2 \vw \).

Could \( c_2 = 0 \)? No because then \( \vu \) would be a scalar multiple of \( \vv \), which is a contradiction.

So we get \( \vw = (1/c_2) \vu - (c_1/c_2) \vv \), hence \( \vw \) is a linear combination of \( \vu \) and \( \vv \), meaning that \( \vw \) is in \( \text{Span}(\vu, \vv) \).
There are some situations where we can guarantee the linear dependence of vectors.

**Theorem 8.** For $p$ vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ in $\mathbb{R}^n$, if $p > n$, then the $p$ vectors are linearly dependent.

**Proof.** In row reducing the $n \times p$ matrix $A = [\vec{v}_1 \ \vec{v}_2 \ \cdots \ \vec{v}_p]$ there are more columns than pivot positions.

Thus there are free variables and hence nontrivial solutions of $A\vec{x} = \vec{0}$, meaning the columns of $A$ are linearly dependent. \hfill $\square$

**Theorem 9.** If one of the $p$ vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ in $\mathbb{R}^n$ is the zero vector, then the set of $p$ vectors is linearly dependent.

**Proof.** In the linear combination $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p$ with WLOG say $\vec{v}_p = \vec{0}$, we can take the first $p-1$ weights all to be zero while taking $c_p = 1$, thus making the linear combination sum to $\vec{0}$. \hfill $\square$