Vector Spaces. Many sets of mathematical objects have the algebraic properties that $\mathbb{R}^n$ does. They have a scalar multiplication and an addition that behave just like those of $\mathbb{R}^n$.

The idea of a vector space is a unifying principle in that properties of the vectors depend only on the properties of the scalar multiplication and addition but not on the elements of the vector space.

Definition. A vector space is a set $V$ of objects denoted as $\vec{u}, \vec{v}, \vec{w}$, etc., on which are defined two operations, scalar multiplication $\alpha \vec{u}$ and addition $\vec{u} + \vec{v}$, which operations satisfy the following axioms:

1. the sum $\vec{u} + \vec{v}$ is in $V$ for all $\vec{u}, \vec{v}$ in $V$,
2. $\vec{x} + \vec{y} = \vec{y} + \vec{x}$ for all $\vec{x}, \vec{y} \in V$,
3. $(\vec{x} + \vec{y}) + \vec{w} = \vec{x} + (\vec{y} + \vec{w})$ for all $\vec{x}, \vec{y}, \vec{w} \in V$,
4. There exists an element $\vec{0}$ in $V$ for which $\vec{x} + \vec{0} = \vec{x}$ for each $\vec{x} \in V$,
5. For each $\vec{x} \in V$ there is an element $-\vec{x}$ such that $\vec{x} + (-\vec{x}) = \vec{0}$,
6. the scalar multiple $\alpha \vec{u}$ is in $V$ for all $\alpha$ and for all $\vec{u}$ in $V$,
7. $\alpha(\vec{x} + \vec{y}) = \alpha\vec{x} + \alpha\vec{y}$ for all $\alpha \in \mathbb{R}$ and all $\vec{x}, \vec{y} \in V$,
8. $(\alpha + \beta)\vec{x} = \alpha\vec{x} + \beta\vec{x}$ for all $\alpha, \beta \in \mathbb{R}$ and all $\vec{x} \in V$,
9. $(\alpha\beta)\vec{x} = \alpha(\beta\vec{x})$ for all $\alpha, \beta \in \mathbb{R}$ and all $\vec{x} \in V$,
10. $1\vec{x} = \vec{x}$ for all $\vec{x} \in V$.

Using ONLY these properties, we can show that the vector $\vec{0}$ of Axiom 4 is unique, that the negative $-\vec{u}$ of Axiom 5 is unique, that $0\vec{u} = \vec{0}$, that $\alpha\vec{0} = \vec{0}$, and that $-\vec{u} = (-1)\vec{u}$.

The Vector Space $\mathbb{R}^{m \times n}$. The set of all $m \times n$ matrices is denoted by $\mathbb{R}^{m \times n}$.

The set $\mathbb{R}^{m \times n}$ has a scalar multiplication and addition defined on it: if $A = (a_{ij})$, $B = (b_{ij})$ are $m \times n$ matrices, and $\alpha$ is a scalar, then

$$\alpha A = (\alpha a_{ij}) \quad \text{and} \quad A + B = (a_{ij} + b_{ij}).$$

The algebraic operations of scalar multiplication and of addition obey the ten axioms of a vector space: these rules are nothing more than the rules of matrix algebra.

The Vector Space $C[a,b]$. Now for a set of “objects” you may not have thought of as a vector space.

Let $[a,b]$ be a closed interval with finite nonzero length, i.e., $-\infty < a < b < \infty$.

Let $C[a,b]$ denote the set of functions continuous on $[a,b]$ (where continuity at the endpoints is understood as one-sided).
On $C[a,b]$ there is a scalar multiplication: if $\alpha \in \mathbb{R}$ and $f \in C[a,b]$, then the scalar multiple of $f$ by $\alpha$ is the continuous function

$$(\alpha f)(x) = \alpha f(x), \; x \in [a,b].$$

On $C[a,b]$ there is an addition: if $f$ and $g$ are in $C[a,b]$, then the sum of $f$ and $g$ is the continuous function

$$(f + g)(x) = f(x) + g(x), \; x \in [a,b].$$

Axioms 2 and 3 are satisfied here:

$$(f + g)(x) = f(x) + g(x) = g(x) + f(x) = (g + f)(x),$$

$$(f + g)(x) + h(x) = f(x) + g(x) + h(x) = f(x) + (g + h)(x).$$

What is the zero vector in $C[a,b]$? It is the zero function,

$$z(x) = 0 \text{ for all } z \in [a,b].$$

It satisfies $(f + z)(x) = f(x)$. Thus axiom 4 is satisfied.

The rest of the axioms are left to you to verify.

The Vector Space $\mathbb{P}_n$. Let $\mathbb{P}_n$ denote the set of all polynomials of degree at most $n$, having real coefficients.

Scalar multiplication and addition on $\mathbb{P}_n$ are defined like those on $C[a,b]$:

$$(\alpha p)(x) = \alpha p(x), \quad (p + q)(x) = p(x) + q(x).$$

The scalar multiple of a polynomial of degree at most $n$, with real coefficients, is a polynomial of degree at most $n$, with real coefficients.

The addition of two polynomials of degree at most $n$, with real coefficients, is a polynomial of degree at most $n$, with real coefficients.

Thus these algebraic operations on $\mathbb{P}_n$ satisfy axioms 1 and 6.

What is the zero vector in $\mathbb{P}_n$? It is the zero polynomial,

$$z(x) = 0x^n + \cdots + 0x + 0.$$

It is left to you to verify that the other axioms are satisfied.

Subspaces. Certain subsets of a vector space $V$ are themselves vector spaces.

To detect which subsets $H$ of $V$ are subspaces requires verifying only three of the axioms in $H$, because the rest following from the axioms of $V$.

Definition. A **subspace** of a vector space $V$ is a subset $H$ for which

a. $H$ contains the zero vector $\vec{0}$ of $V$,

b. $H$ is closed under addition, i.e., for every $\vec{u}$ and $\vec{v}$ are in $H$ we have $\vec{u} + \vec{v}$ is in $H$,

c. $H$ is closed under scalar multiplication, i.e., for every $\vec{u}$ in $H$ and every scalar $\alpha$, we have $\alpha \vec{u}$ is in $H$. 
Example. Let $S = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in \mathbb{R}^{2 \times 2} : a_{11} + a_{22} = 0 \right\}$.

We check the three conditions in the definition of a subspace.

The zero $2 \times 2$ matrix $0$ satisfies $a_{11} + a_{22} = 0$, and so $0 \in S$.

If $\alpha \in \mathbb{R}$, $A = (a_{ij}) \in S$, and $B = (b_{ij}) \in S$, then

$$\alpha A = \begin{bmatrix} \alpha a_{11} & \alpha a_{12} \\ \alpha a_{21} & \alpha a_{22} \end{bmatrix}$$

satisfies $\alpha a_{11} + \alpha a_{22} = \alpha(a_{11} + a_{22}) = 0$, and

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix}$$

satisfies $(a_{11} + b_{11}) + (a_{22} + b_{22}) = (a_{11} + a_{22}) + (b_{11} + b_{22}) = 0$.

Thus $S$ is a subspace of $\mathbb{R}^{2 \times 2}$.

Example. Let $\mathbb{P}$ be the set of all polynomials with real coefficients.

The subset $\mathbb{P}_n$ of polynomials of degree at most $n$ is a subspace of $\mathbb{P}$.

Example. Is the subset

$$H = \left\{ \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} : s, t \in \mathbb{R} \right\}$$

a subspace of $\mathbb{R}^3$?

Well, $H$ contains $\vec{0}$, scalar multiples of elements of $H$ are in $H$, and the sum of any two vectors in $H$ is in $H$, and so $H$ is a subspace of $\mathbb{R}^3$.

Is $H$ just $\mathbb{R}^2$? It isn’t. Why not?

Example. Let $L$ be a line in the plane $\mathbb{R}^2$ that does not pass through the origin.

Is $L$ a subspace of $\mathbb{R}^2$? No, why not? It does not contain $\vec{0}$, scalar multiples of vectors in $L$ are not in $L$, and sums of vectors in $L$ are not in $L$.

Example. Let $S = \{ p(x) \in \mathbb{P}_5 : p(1) + p'(1) = 0 \}$, the subset of those polynomials of degree at most 5 for which $p(1)$ plus its derivative $p'(1)$ equals 0.

The zero polynomial $z(x)$ satisfies $z(1) + z'(1) = 0$, and so $z(x) \in S$.

If $\alpha \in \mathbb{R}$, $p(x) \in S$, and $q(x) \in S$, then $\alpha p(x)$ satisfies

$$\alpha p(1) + \alpha p'(1) = \alpha (p(1) + p'(1)) = 0,$$

and $(p + q)(x)$ satisfies

$$(p(1) + q(1)) + (p'(1) + q'(1)) = (p(1) + p'(1)) + (q(1) + q'(1)) = 0.$$

Thus the subset $S$ is a subspace of $\mathbb{P}_5$. 
A Subspace Spanned by a Set. A common way of describing a subspace is by the means of linear combinations.

**Theorem 1.** Let \( V \) be a vector space and \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \) vectors in \( V \). Then the set \( H = \text{Span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\} \) is a subspace of \( V \).

**Proof.** The “trivial” linear combination

\[
0\vec{v}_1 + 0\vec{v}_2 + \cdots + 0\vec{v}_k = \vec{0} \in H.
\]

For any scalar \( \alpha \) and any \( \vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k \in H \), we have

\[
\alpha \vec{v} = (\alpha c_1)\vec{v}_1 + (\alpha c_2)\vec{v}_2 + \cdots + (\alpha c_k)\vec{v}_k \in H,
\]

For two vectors \( \vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k \) and \( \vec{u} = d_1\vec{v}_1 + d_2\vec{v}_2 + \cdots + d_k\vec{v}_k \) we have

\[
\vec{v} + \vec{u} = (c_1 + d_1)\vec{v}_1 + (c_2 + d_2)\vec{v}_2 + \cdots + (c_k + d_k)\vec{v}_k \in H.
\]

Thus \( H \) is a subspace of \( V \). \( \square \)

[Notice how these calculations have nothing to do with what the actual vectors in \( V \) are, but only with the algebraic properties of scalar multiplication and addition.]

We call \( \text{Span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\} \) the **subspace spanned** (or **generated**) by \( \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\} \).

A **spanning** (or **generating**) set for a subspace \( H \) of \( V \) is a set \( \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\} \) for which \( H = \text{Span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\} \).

**Example.** For an \( m \times n \) matrix \( A \), suppose that the solution set \( H \) of \( A\vec{x} = \vec{0} \) has the form \( \vec{x} = s\vec{a} + t\vec{b} \).

Then the solution set \( H \) is the subspace of \( \mathbb{R}^n \) spanned by \( \{\vec{a}, \vec{b}\} \).