There are several subspaces associated with an $m \times n$ matrix $A$ that are useful in analyzing $A$.

Having used these subspaces, we give them names.

**Definition.** The **null space** of an $m \times n$ matrix $A$ is the set of all the solutions of the homogeneous equation $A\vec{x} = \vec{0}$, which we denote by

$$\text{Nul}(A) = \{ \vec{x} \in \mathbb{R}^n : A\vec{x} = \vec{0} \}.$$

**Theorem 2.** The null space of an $m \times n$ matrix is a subspace of $\mathbb{R}^n$.

**Proof.** We have three conditions to verify about the set Nul($A$).

First, does Nul($A$) contain $\vec{0}$? Yes, because $A\vec{0} = \vec{0}$.

Second, if $\vec{x} \in \text{Nul}(A)$, does $\alpha \vec{x}$ belong to Nul($A$) for all $\alpha$? Yes, because

$$A(\alpha \vec{x}) = \alpha (A\vec{x}) = \alpha \vec{0} = \vec{0}.$$

Last, if $\vec{x}$ and $\vec{y}$ belong to Nul($A$), does $\vec{x} + \vec{y}$ belong to Nul($A$)? Yes, because

$$A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y} = \vec{0} + \vec{0} = \vec{0}.$$

Thus Nul($A$) is a subspace of $\mathbb{R}^n$. \hfill \square

**Example.** Find the null space of

$$A = \begin{bmatrix} 1 & 2 & -1 & 2 \\ 2 & -1 & 3 & 4 \\ 3 & 1 & 2 & 6 \end{bmatrix}.$$ 

The reduced row echelon form of $A$ is

$$A \sim \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$ 

The variables $x_1$ and $x_2$ are basic, and the variables $x_3$ and $x_4$ are free:

$$x_1 = -x_3 - 2x_4,$$

$$x_2 = x_3.$$

Let $x_3 = t$ and $x_4 = s$. The solution set of $A\vec{x} = \vec{0}$ is

$$\text{Nul}(A) = \left\{ \begin{bmatrix} -t - 2s \\ t \\ t \\ s \end{bmatrix} : t, s \in \mathbb{R} \right\} = \left\{ t \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} : t, s \in \mathbb{R} \right\}.$$
The null space of \( A \) is the span of two vectors in \( \mathbb{R}^4 \).

Are these two vector linearly independent? Yes, they are, because of different entries where the 0’s and the 1’s appear.

This pattern for the null space of a matrix holds whenever there are 1 or more free variables: we get a linearly independent set of vectors whose span is the null space, and the number of vectors in the spanning set is the number of free variables.

The Column Space of a Matrix. Another subspace associated with a matrix is its column space.

**Definition.** For an \( m \times n \) matrix \( A \), the **column space** of \( A \) is the subset of all linear combinations of the columns of \( A \).

**Theorem 3.** The column space of an \( m \times n \) matrix \( A \) is a subspace of \( \mathbb{R}^m \).

**Proof.** We recognize that for \( A = [\vec{a}_1 \ \vec{a}_2 \ \cdots \ \vec{a}_n] \) we have

\[
\text{Col}(A) = \text{Span}\{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n\},
\]

and so it follows that \( \text{Col}(A) \) is subspace of \( \mathbb{R}^m \). \( \square \)

Other ways to view the column space of a matrix are as

\[
\text{Col}(A) = \{\vec{b} \in \mathbb{R}^n : A\vec{x} = \vec{b} \text{ for some } \vec{x} \in \mathbb{R}^n\}.\]

or as the range of the linear transformation \( \vec{x} \mapsto A\vec{x} \).

Comparing and Contrasting the Null Space and Column Space of a Matrix.

We have now associated to an \( m \times n \) matrix \( A \) the two subspaces \( \text{Nul}(A) \) and \( \text{Col}(A) \).

Which one is a subspace of the domain \( \mathbb{R}^n \)? (\( \text{Nul}(A) \)) of the codomain \( \mathbb{R}^m \)? (\( \text{Col}(A) \))

Which one is easier to compute? The column space is because it is the span of the columns of \( A \), while to find the null space we have to row reduce \( A \).

When does \( \text{Nul}(A) = \{\vec{0}\} \) (the trivial subspace of \( \mathbb{R}^n \))? When \( A\vec{x} = \vec{0} \) has only the trivial solution.

When does \( \text{Col}(A) = \mathbb{R}^m \)? When \( A\vec{x} = \vec{b} \) is consistent for all \( \vec{b} \in \mathbb{R}^n \).

There are more of these comparisons and contrasts in the text.

Generalization of Linear Transformations. Previous for a linear transformation, we set its domain to be the vector space \( \mathbb{R}^n \) and its codomain to be the vector space \( \mathbb{R}^m \).

We can generalize the definition of the linear transformation to a domain that is a vector space and a codomain that is a vector space.

**Definition.** Let \( V \) and \( W \) be vector spaces. A transformation \( T : V \to W \) is a rule that assigns to each \( \vec{v} \) in \( V \) a vector \( T(\vec{v}) \) in \( W \). A transformation \( T : V \to W \) is linear if

1. \( T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v}) \) for all \( \vec{u}, \vec{v} \in V \), and
2. \( T(\alpha \vec{v}) = \alpha T(\vec{v}) \) for all \( \alpha \in \mathbb{R} \) and all \( \vec{v} \in V \).
We can also extend the notions of null space and column space of a linear transformation \( \vec{x} \mapsto A\vec{x} \) to a linear transformation \( T : V \to W \).

**Definition.** The **kernel** of a linear transformation is the set of all \( \vec{v} \) in \( V \) such that \( T(\vec{v}) = \vec{0} \).

Is the kernel of a linear transformation a subspace of \( V \)?

Yes it is because \( T(\vec{0}) = T(0\vec{0}) = 0T(\vec{0}) = \vec{0} \), and for \( \vec{u} \) and \( \vec{v} \) in the kernel of a linear \( T \) we have

\[
T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v}) = \vec{0} + \vec{0}, \quad T(\alpha \vec{u}) = \alpha T(\vec{u}) = \alpha \vec{0} = \vec{0}.
\]

Recall the **range** of a transformation \( T : V \to W \) (not assumed linear) is the set of vectors \( \vec{w} \) in \( W \) for which there exists \( \vec{v} \) in \( V \) such that \( T(\vec{v}) = \vec{w} \).

If \( T \) is linear, if the range of \( T \) a subspace of \( W \)?

Yes it is because for \( \vec{w} \) and \( \vec{z} \) in the range of a linear \( T \) there are \( \vec{u} \) and \( \vec{v} \) in \( V \) for which \( T(\vec{u}) = \vec{w} \) and \( T(\vec{v}) = \vec{v} \), so that

\[
\vec{w} + \vec{z} = T(\vec{u}) + T(\vec{v}) = T(\vec{u} + \vec{v}), \quad \alpha \vec{w} = \alpha T(\vec{u}) = T(\alpha \vec{u}),
\]

saying that \( \vec{w} + \vec{z} \) and \( \alpha \vec{w} \) are in the range of \( T \).

For the range to be subspace, the zero vector needs to be in it. Is \( \vec{0} \) in the range of a linear \( T \)?

Yes it is because \( T(\vec{0}) = T(0\vec{0}) = 0T(\vec{0}) = \vec{0} \).

**Example** Let \( V = C^1[a,b] \) be the vector space of all real valued function defined on \([a,b]\) such that they are differentiable with continuous derivatives.

Let \( W = C[a,b] \), the vector space of continuous real valued functions defined on \([a,b]\).

Is the transformation \( D : V \to W \) defined by \( D(f) = f' \) linear?

For \( f \) and \( g \) in \( V \) we have

\[
D(f + g) = (f + g)' = f' + g', \quad D(\alpha f)(\alpha f)' = \alpha f',
\]

so indeed \( D \) is linear.

What is the kernel of \( D \)? The subspace of all of the constant functions.

What is the range? It is all of \( W \) by the Fundamental Theorem of Calculus which says that any continuous function \( h \) on \([a,b]\) is the derivative of a function with a continuous derivative.