Now that we know how to find a basis of a vector space or its subspaces, we will learn why a basis is important.

We are familiar with the \((x, y)\)-coordinates of the plane: the \(x\) and \(y\) quantities represent horizontal and vertical displacements from a starting point (we call the origin).

A basis of a vector space is the mechanism by which we impose coordinates on a vector space.

The following result is the key to this mechanism.

**Theorem 7 (The Unique Representation Theorem).** Let \(B = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}\) be a basis for a vector space \(V\). Then for each \(\vec{x}\) in \(V\) there exists a unique set of weights \(c_1, c_2, \ldots, c_n\) such that

\[
\vec{x} = c_1 \vec{b}_1 + c_2 \vec{b}_2 + \cdots + c_n \vec{b}_n.
\]

**Proof.** Let \(\vec{x}\) in \(V\) be arbitrary.

Since \(B\) spans \(V\), there exists a set of weights \(c_1, c_2, \ldots, c_n\) such that

\[
\vec{x} = c_1 \vec{b}_1 + c_2 \vec{b}_2 + \cdots + c_n \vec{b}_n.
\]

To show the uniqueness of the set of weights, we suppose there is another set of weights \(d_1, d_2, \ldots, d_n\) such that

\[
\vec{x} = d_1 \vec{b}_1 + d_2 \vec{b}_2 + \cdots + d_n \vec{b}_n.
\]

Subtracting the one linear combination for \(\vec{b}\) from the other gives

\[
0 = (c_1 - d_1) \vec{b}_1 + (c_2 - d_2) \vec{b}_2 + \cdots + (c_n - d_n) \vec{b}_n.
\]

Linear independence of \(B\) implies that the weights in this linear combination are all zero, so that \(c_i = d_i\) for all \(i = 1, 2, \ldots, n\).

Thus there is a unique set of weights \(c_1, c_2, \ldots, c_n\) the representation of \(\vec{x}\) in terms of the basis \(B\) is unique. \(\square\)

Let us review a familiar example of basis and coordinates.

**Example.** For the standard basis \(\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}\) of \(\mathbb{R}^n\), the unique representation of \(\vec{x}\) in \(\mathbb{R}^n\) is

\[
\vec{x} = x_1 \vec{e}_1 + x_2 \vec{e}_2 + \cdots + x_n \vec{e}_n
\]

where the unique weights \(x_1, x_2, \ldots, x_n\) are the **standard coordinates** of \(\vec{x}\).

We can extract idea of standard coordinates and apply it to any basis of any vector space.

**Definition.** Suppose \(B = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}\) is a basis for a vector space \(V\). The **coordinates of \(\vec{x}\) relative to the basis \(B\)** (or the **B-coordinates of \(\vec{x}\)**) are the unique weights \(c_1, c_2, \ldots, c_n\) for which

\[
\vec{x} = c_1 \vec{b}_1 + c_2 \vec{b}_2 + \cdots + c_n \vec{b}_n.
\]
Notationally we write and say that
\[
[x]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}
\]
is the coordinate vector of \( \vec{x} \) (relative to \( B \)), or the \( B \)-coordinate vector of \( \vec{x} \).

Example. The set
\[
B = \{ \vec{b}_1, \vec{b}_2 \} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}
\]
form a basis for \( \mathbb{R}^2 \) because the matrix \( A = [\vec{b}_1, \vec{b}_2] \) is invertible (which by the Inverse Matrix Theorem says that the columns span and are linearly independent).

What are the \( B \)-coordinates of \( \vec{e}_1 \)? We are asked to solve the vector equation
\[
c_1 \vec{b}_1 + c_2 \vec{b}_2 = \vec{e}_1
\]
or in other notation, the matrix equation \( A \vec{c} = \vec{e}_1 \).

This we do by row reduction of the augmented matrix to obtain \( c_1 = 1/2 \) and \( c_2 = 1/2 \), so that
\[
[\vec{e}_1]_B = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}.
\]

The basis \( B \) gives a different “view” of the plane, in which \( \vec{b}_1 \) and \( \vec{b}_2 \) correspond to the dials a different kind of Etch-A-Sketch (one dial moves northeast-southwest, the other dial moves southeast-northwest).

There is an important aspect of coordinates that is implicit in the above example that we extract.

In finding the coordinates of \( \vec{x} \) in \( \mathbb{R}^n \) relative to a basis \( B = \{ \vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n \} \), we solved the equation
\[
c_1 \vec{b}_1 + c_2 \vec{b}_2 + \cdots + c_n \vec{b}_n = \vec{x}.
\]
If we set \( P_B \) to be the matrix \( \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_n \end{bmatrix} \), then the above vector equation in matrix notation is
\[
\vec{x} = P_B [\vec{x}]_B.
\]

We call the matrix \( P_B \) the change-of-coordinates matrix from the basis \( B \) to the standard basis of \( \mathbb{R}^n \).

The matrix \( P_B \) is invertible (why?) and gives the change of coordinates from the standard basis to the basis \( B \).

The Coordinate Mapping. The choice of a basis \( B \) for a vector space \( V \) gives a transformation \( \vec{x} \mapsto [\vec{x}]_B \) from the possible unfamiliar vector space \( V \) (the domain) to the familiar vector space \( \mathbb{R}^n \) (the codomain).
What are the properties of this transformation? Is it linear? Is it one-to-one? Is it onto?

**Theorem 8.** For a basis \( \mathbf{B} = \{ \vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n \} \) of a vector space \( V \), the coordinate mapping \( \vec{x} \mapsto [\vec{x}]_\mathbf{B} \) is a one-to-one linear transformation from \( V \) onto \( \mathbb{R}^n \).

**Proof.** To show that the coordinate mapping is linear, suppose there are two vectors in \( V \), say

\[
\vec{u} = c_1 \vec{b}_1 + c_2 \vec{b}_2 + \cdots + c_n \vec{b}_n,
\]
\[
\vec{v} = d_1 \vec{b}_1 + d_2 \vec{b}_2 + \cdots + d_n \vec{b}_n.
\]

Then

\[
\vec{u} + \vec{v} = (c_1 + d_1) \vec{b}_1 + (c_2 + d_2) \vec{b}_2 + \cdots + (c_n + d_n) \vec{b}_n.
\]

It follows that

\[
[\vec{u} + \vec{v}]_\mathbf{B} = \begin{bmatrix} c_1 + d_1 \\ c_2 + d_2 \\ \vdots \\ c_n + d_n \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} + \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix} = [\vec{u}]_\mathbf{B} + [\vec{v}]_\mathbf{B}.
\]

For a scalar \( \alpha \) we have

\[
\alpha \vec{u} = \alpha c_1 \vec{b}_1 + \alpha c_2 \vec{b}_2 + \cdots + \alpha c_n \vec{b}_n,
\]

and so

\[
[\alpha \vec{u}]_\mathbf{B} = \begin{bmatrix} \alpha c_1 \\ \alpha c_2 \\ \vdots \\ \alpha c_n \end{bmatrix} = \alpha \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \alpha [\vec{u}]_\mathbf{B}.
\]

Thus we have shown that the coordinate mapping is linear.

The proofs of the one-to-one and the onto are homework problems (#23,24).

The one-to-one linear coordinate mapping \( \vec{x} \mapsto [\vec{x}]_\mathbf{B} \) from a vector space \( V \) with basis \( \mathbf{B} \) onto \( \mathbb{R}^n \) is an example of an important type of transformation called an *isomorphism*.

It means that the possibly less familiar vector space \( V \) is just a “copy” of the more familiar \( \mathbb{R}^n \): every vector space calculation in \( V \) (i.e., linear combinations) is accurately reproduced in \( \mathbb{R}^n \).

**Example.** We have seen that the set of polynomials \( \mathbf{B} = \{ 1, t, \ldots, t^n \} \) is a basis for the less familiar vector space \( \mathbb{P}_n \).

The coordinate mapping \( p(t) \mapsto [p(t)]_\mathbf{B} \) is the isomorphism

\[
c_0 1 + c_1 t + \cdots + c_n t^n \mapsto \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix}.
\]

The image of the basis vector 1 is the vector \( \vec{e}_1 \), the image of the basis vector \( t \) is \( \vec{e}_2 \), which continues, ending with the image of the basis vector \( t^n \) is \( \vec{e}_{n+1} \).

So \( \mathbb{P}_n \) is nothing more than the vector space \( \mathbb{R}^{n+1} \) in disguise!