Recall that the coordinate mapping is an isomorphism from a vector space with a basis of \( n \) vectors onto \( \mathbb{R}^n \).

Could there be a different basis of \( V \) that had more or less than \( n \) vectors?

**Theorem 9.** If a vector space \( V \) has a basis \( \mathcal{B} = \{ \vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n \} \) of \( n \) vectors, then any set of vectors in \( V \) containing more than \( n \) vectors is linearly dependent.

**Proof.** Suppose \( \{ \vec{u}_1, \vec{u}_2, \ldots, \vec{u}_p \} \) is a set of vectors in \( V \) with \( p > n \).

The \( p \) coordinate vectors \( [\vec{u}_1]_{\mathcal{B}}, [\vec{u}_2]_{\mathcal{B}}, \ldots, [\vec{u}_p]_{\mathcal{B}} \) are linearly dependent in \( \mathbb{R}^n \) because \( p > n \) (more columns than rows).

So there are weights \( c_1, c_2, \ldots, c_p \), not all zero, such that

\[
c_1[\vec{u}_1]_{\mathcal{B}} + c_2[\vec{u}_2]_{\mathcal{B}} + \cdots + c_p[\vec{u}_p]_{\mathcal{B}} = \vec{0}.
\]

Since the coordinate mapping is linear we have that

\[
[c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_p\vec{u}_p]_{\mathcal{B}} = \vec{0}.
\]

Since the coordinate mapping is one-to-one we obtain that

\[
c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_p\vec{u}_p = \vec{0}.
\]

With at least one of the weights \( c_1, c_2, \ldots, c_p \) not zero, we conclude that \( \{ \vec{u}_1, \vec{u}_2, \ldots, \vec{u}_p \} \) is linearly dependent.

An immediate consequence of Theorem 9 is that any basis of \( V \) cannot have more than \( n \) vectors in it.

Could a basis of \( V \) have less than \( n \) vectors?

**Theorem 10.** If a vector space \( V \) has a basis of \( n \) vectors, then every basis of \( V \) has \( n \) vectors.

**Proof.** Let \( \mathcal{B}_1 \) be a basis of \( n \) vectors, and let \( \mathcal{B}_2 \) be an arbitrary basis of \( V \).

If \( \mathcal{B}_2 \) has more than \( n \) vectors, then by Theorem 9, the set \( \mathcal{B}_2 \) is linearly dependent and cannot be a basis.

So \( \mathcal{B}_2 \) has no more than \( n \) vectors in it.

Now suppose that the basis \( \mathcal{B}_2 \) has less than \( n \) vectors in it.

Again by Theorem 9, the set \( \mathcal{B}_1 \) would be linearly dependent and cannot be a basis.

So \( \mathcal{B}_2 \) has no less than \( n \) vectors in it.

Thus the basis \( \mathcal{B}_2 \) has exactly \( n \) vectors in it. \( \square \)

Once we know that a nonzero vector space has a basis with a finite number of vectors in it, then every basis of that vector space has the same number of vectors in it.
We can show that a nonzero vector space has a basis with a finite number of vectors in it, by finding a finite spanning set $S$ for the vector space (since by the Spanning Set Theorem, a subset of $S$ is a basis).

We give the number of vectors common to all bases of a vector space a name.

Definitions. We say that a vector space $V$ is spanned by a finite set is **finite-dimensional**, and the **dimension** of $V$, written as $\dim V$, is the number of vectors in any basis of $V$.

The dimension of the zero vector space $\{\vec{0}\}$ is $\dim V = 0$.

A vector space not spanned by a finite number of vectors in it, is said to be **infinite-dimensional**.

Examples. (a) The standard basis $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$ of $\mathbb{R}^n$ contains $n$ vectors, and so $\dim \mathbb{R}^n = n$.

(b) A basis for $\mathbb{P}_n$ is the set of $n+1$ vectors $\{1, t, \ldots, t^n\}$, and so $\dim \mathbb{P}_n = n + 1$.

(c) What is the dimension of $\mathbb{R}^{2 \times 2}$ (or what the text denotes $M_{2 \times 2}$)?

For an arbitrary $2 \times 2$ matrix $A$ we have

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$  

We expressed every $2 \times 2$ matrix as a linear combination of 4 matrices (the span of four matrices).

Are the four matrices linearly independent?

Yes, they are, because setting to the zero matrix the linear combination of the four matrices with weights $a, b, c, d$ shows that $a = b = c = d = 0$.

Thus the four matrices are a basis $\mathcal{B}$ for $\mathbb{R}^{2 \times 2}$, and so $\dim \mathbb{R}^{2 \times 2} = 4$.

What is the coordinate mapping from $\mathbb{R}^{2 \times 2}$ to $\mathbb{R}^4$ for this basis? It is

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto [A]_\mathcal{B} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}.$$  

The vector space $\mathbb{R}^{2 \times 2}$ is just $\mathbb{R}^4$ in disguise.

(d) Is the vector space $\mathbb{P}$ of all polynomials finite dimensional?

It is not spanned by the linearly independent set $\{1, t, \ldots, t^n\}$ for each choice of $n$ because $t^{n+1}$ is not in its span.

Thus $\mathbb{P}$ is infinite dimensional.

(e) Is the vector space $C[a, b]$ finite dimensional?
It is not spanned by the linearly independent set \( \{1, t, \ldots, t^n\} \) (with their domains restricted to \([a, b]\)) for each \( n \) because \( t^{n+1} \) (with domain restricted to \([a, b]\)) is not in its span.

Thus \( C[a, b] \) is infinite dimensional.

What is the relationship between the dimension of a vector space \( V \) and the dimension of a subspace \( H \) of \( V \)?

**Theorem 11.** Let \( H \) be a subspace of a finite dimensional vector space \( V \). Any linearly independent set in \( H \) can be expanded, if needed, to a basis of \( H \). Also, \( H \) is finite dimensional and \( \dim H \leq \dim V \).

**Proof.** If \( H = \{\vec{0}\} \), then \( \dim H = 0 \leq \dim V \).

So suppose \( H \neq \{\vec{0}\} \), and let \( S = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\} \) be a linearly independent set in \( H \).

If \( S \) spans \( H \), then \( S \) is a basis for \( H \).

If \( S \) does not span \( H \), then there is a vector \( \vec{u}_{k+1} \) in \( H \) but not in the span of \( S \).

The expansion of \( S \) obtained by adding \( \vec{u}_{k+1} \) to it results in the new set \( \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k, \vec{u}_{k+1}\} \), which is linearly independent because no vector in it is a linear combination of the vectors that precede it (Theorem 4).

If the new set does not span \( H \), we continue to add vectors until we obtain a linearly independent set that spans \( H \).

The number of vectors in the expansion of \( S \) cannot exceed the dimension of \( V \), because any set of vectors with more than \( \dim V \) vectors in it is linearly dependent (by Theorem 9).

We therefore obtain that \( \dim H \leq \dim V \). \( \square \)

When we know the dimension \( p \) of a finite dimensional vector space (or a nonzero subspace of it), the search for a basis for the vector space need only check one of the linear independence of \( p \) vectors or the span of \( p \) vectors (but not both) to get a basis.

**Theorem 12** (The Basis Theorem). Let \( V \) be a \( p \)-dimensional vector space with \( p \geq 1 \). Any linearly independent set of \( p \) vectors in \( V \) is a basis for \( V \), and any set of \( p \) vectors that span \( V \) is a basis for \( V \).

**Proof.** If a linearly independent set of \( p \) vectors did not span \( V \), then applying Theorem 10 with \( H = V \), we can enlarge the linearly independent set of \( p \) vectors to a linearly independent set of \( p + 1 \) vectors in \( V \).

But \( V \) is \( p \)-dimensional, and any set of \( p+1 \) vectors in it is linearly dependent by Theorem 9.

This contradiction implies that the set of \( p \) linearly independent vectors does span \( V \), and is hence a basis.

On the other hand, suppose a set \( S \) of \( p \) vectors spans \( V \).

By the Spanning Set Theorem, a subset \( S' \) of \( S \) is a basis for \( V \).
The number of vectors in the basis $S'$ is $p$ by Theorem 10 because $\dim V = p$.
The only subset of $S'$ with $p$ elements is $S$, and so $S' = S$, meaning that $S$ is a basis for $V$. \qed