We have learned many things about solving the matrix equation \( A\vec{x} = \vec{b} \).

Now we turn our attention to solving another problem in linear algebra, namely the equation
\[
A\vec{x} = \lambda\vec{x}
\]
for an \( n \times n \) matrix \( A \) and a scalar \( \lambda \).

**Example.** For the matrix
\[
A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}
\]
and the scalar \( \lambda = 3 \), the vector
\[
\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}
\]
satisfies \( A\vec{x} = \lambda\vec{x} \) because
\[
A\vec{x} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} \quad \text{and} \quad \lambda\vec{x} = 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}.
\]

Are there other choices of \( \vec{x} \) and \( \lambda \) that work too?

Certainly the choice of \( \vec{x} = \vec{0} \) works for any scalar \( \lambda \), but this does not depend on what \( A \) is, so we will exclude this *trivial case*.

**Definitions.** An **eigenvector** of an \( n \times n \) matrix \( A \) is a NONZERO vector \( \vec{x} \) such that \( A\vec{x} = \lambda\vec{x} \) for some scalar \( \lambda \).

An **eigenvalue** of an \( n \times n \) matrix \( A \) is a scalar (could be zero) such that there is a NONZERO vector \( \vec{x} \) such that \( A\vec{x} = \lambda\vec{x} \); such a vector is called an eigenvector corresponding to \( \lambda \).

What is the significance of the eigenvectors and eigenvalues of \( A \)?

**Example (Continued).** We know that \( \lambda = 3 \) is an eigenvalue of the matrix
\[
A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix},
\]
and a corresponding eigenvector is
\[
\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.
\]

The eigenvector satisfies the equation \( A\vec{x} = 3\vec{x} \), or rewritten,
\[
A\vec{x} - 3\vec{x} = \vec{0}.
\]

We would like to factor out the common \( \vec{x} \), but we might mistakenly write
\[
(A - 3)\vec{x} = \vec{0}.
\]
What is incorrect with this equation? There is no way to subtract the scalar $3$ from the $2 \times 2$ matrix $A$.

Instead, we write $3\vec{x} = 3I\vec{x}$ for the $2 \times 2$ identity matrix $I$, so that

$$(A - 3I)\vec{x} = \vec{0}.$$ 

This homogeneous equation has a nontrivial solution in $\vec{x}$.

By the Inverse Matrix Theorem the matrix $A - 3I$ is not invertible, which we verify directly: the determinant of

$$A - 3I = \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix}$$

is 0, and so $A - 3I$ is not invertible.

The nonzero vector $\vec{x}$ belongs to the null space of $A - 3I$; in particular, $\vec{x}$ is a basis for Nul$(A - 3I)$ because

$$A - 3I \sim \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$$

so that $u_1$ is a basic variable, $u_2 = t$ is a free variable, and the solution set of $(A - 3I)\vec{u} = \vec{0}$ consists of the vectors

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = t \begin{bmatrix} 1 \\ 2 \end{bmatrix} = t\vec{x}.$$ 

Are there other values of $\lambda$ for which $A - \lambda I$ is not invertible? Yes, $\lambda = -1$.

An eigenvector of $A$ corresponding to $\lambda = -1$ is a nontrivial solution of $(A + I)\vec{u} = \vec{0}$, namely

$$\vec{x} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$ 

It so happens that this $\vec{x}$ is a basis for Nul$(A + I)$.

We see in this example that eigenvectors of $A$ are basis vectors of null spaces of matrices of the form $A - \lambda I$.

Thus $\lambda$ is an eigenvalue of $A$ if and only if the homogeneous equation $(A - \lambda I)\vec{x} = \vec{0}$ has a nontrivial solution.

For an eigenvalue $\lambda$ of $A$ we call the null space of $A - \lambda I$ the **eigenspace** of $A$ corresponding to $\lambda$.

For a real eigenvalue $\lambda$ of a matrix $A$, the matrix transformation $\vec{x} \mapsto A\vec{x} = \lambda\vec{x}$ acts like dilation/contraction on the eigenspace Nul$(A - \lambda I)$.

At this point, we only have the method of guess and check to find eigenvalues of a square matrix.

But for a certain type of square matrix, the eigenvalues are readily found.

**Theorem 1.** The eigenvalues of a triangular square matrix are its diagonal entries.
WARNING: this does not say that we can find eigenvalues by row reduction. A counterexample to this is the invertible matrix

\[
A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}
\]

whose eigenvalues are 3 and -1, but the eigenvalues of the I (the reduced row echelon form of A) are 1 and 1.

Proof. Suppose \( A \) is in triangular form with diagonal entries \( a_{ii} \) for \( i = 1, 2, \ldots, n \).

For a fixed \( i \) set \( \lambda = a_{ii} \).

One of the diagonal entries of \( A - \lambda I \) is zero (namely the \((i, i)\) entry), and so \( \det(A - \lambda I) = 0 \).

By the Inverse Matrix Theorem, there is a nontrivial solution \( \vec{x} \) of \((A - \lambda I)\vec{x} = \vec{0}\).

This means that \( A\vec{x} = \lambda\vec{x} \), and so \( \lambda \) is an eigenvalue. \( \square \)

The eigenvectors of a square matrix associated to distinct eigenvalues enjoy a valuable property.

**Theorem 2.** If \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r \) are eigenvectors corresponding to distinct eigenvalues of a square matrix, then the set \( \{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r \} \) is linearly independent.

Proof. We prove this by way of contradiction, by supposing that the set of eigenvectors is linearly dependent.

Since \( \vec{v}_1 \neq \vec{0} \), we can apply the linear dependent Theorem: there is a least index \( p \) such that the eigenvector \( \vec{v}_{p+1} \) is a linear combination of the preceding \( p \) eigenvectors.

The choice of the least index \( p \) means that the set of eigenvectors \( \{ \vec{v}_1, \ldots, \vec{v}_p \} \) is linearly independent (for otherwise there would be a smaller choice of \( p \)).

There are weights \( c_1, c_2, \ldots, c_p \) such that

\[
c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p = \vec{v}_{p+1}.
\]

To this we apply the matrix transformation \( \vec{x} \mapsto A\vec{x} \) to obtain

\[
c_1A\vec{v}_1 + c_2A\vec{v}_2 + \cdots + c_pA\vec{v}_p = A\vec{v}_{p+1}.
\]

Since \( A\vec{v}_i = \lambda_i\vec{v}_i \) for each \( i = 1, 2, \ldots, p + 1 \), we obtain

\[
c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2 + \cdots + c_p\lambda_p\vec{v}_p = \lambda_{p+1}\vec{v}_{p+1}.
\]

Multiplying the equation \( c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p = \vec{v}_{p+1} \) through by \( \lambda_{p+1} \) gives

\[
c_1\lambda_{p+1}\vec{v}_1 + c_2\lambda_{p+1}\vec{v}_2 + \cdots + c_p\lambda_{p+1}\vec{v}_p = \lambda_{p+1}\vec{v}_{p+1}.
\]

Subtracting this equation from \( c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2 + \cdots + c_p\lambda_p\vec{v}_p = \lambda_{p+1}\vec{v}_{p+1} \) gives

\[
c_1(\lambda_1 - \lambda_{p+1})\vec{v}_1 + c_2(\lambda_2 - \lambda_{p+1})\vec{v}_2 + \cdots + c_p(\lambda_p - \lambda_{p+1})\vec{v}_p = \vec{0}.
\]
The linear independence of the set \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\} now implies that \(c_i(\lambda_i - \lambda_{p+1}) = 0\) for each \(i = 1, 2, \ldots, p\).

Thus \(c_i = 0\) for each \(i = 1, 2, \ldots, p\) since the eigenvalues are distinct.

This implies that \(v_{p+1} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_p \vec{v}_p = 0\).

[What is the contradiction here?]

But \(\vec{v}_{p+1}\) is an eigenvector, meaning it is nonzero, a contradiction. \qed