A Motivational Example. Recall that the eigenvalues and eigenvectors of
\[ A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \text{ are } \lambda_1 = 3, \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \lambda_2 = -1, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}. \]

Let \( P \) be the matrix whose columns are the eigenvectors of \( A \):
\[ P = [\vec{v}_1 \ \vec{v}_2] = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}. \]

The matrix \( P \) is invertible since its columns are linearly independent; its inverse is
\[ P^{-1} = \frac{1}{-4} \begin{bmatrix} -2 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/4 \\ 1/2 & -1/4 \end{bmatrix}. \]

The matrix \( P^{-1}AP \) is similar to the matrix \( A \).

What is \( P^{-1}AP \)? Well,
\[
P^{-1}AP = \begin{bmatrix} 1/2 & 1/4 \\ 1/2 & -1/4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 3/2 & 3/4 \\ -1/2 & 1/4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 3/2 + 3/2 & 3/2 - 3/2 \\ -1/2 + 1/2 & -1/2 - 1/2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}.
\]

Recognize the entries on the diagonal of this matrix? They are the eigenvalues of \( A \) in the order in which their eigenvectors were placed in \( P \)!

Definitions. An \( n \times n \) matrix \( A \) is **diagonalizable** if it is similar to a diagonal matrix.

We call an invertible matrix \( P \) for which \( P^{-1}AP \) is diagonal, a **diagonalizing matrix** for \( A \).

Is every square matrix diagonalizable?

**Theorem 5.** An \( n \times n \) matrix \( A \) is diagonalizable if and only if \( A \) has \( n \) linearly independent eigenvectors.

Proof. For any invertible matrix \( P \) with columns \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \) and any diagonal matrix \( D \) with diagonal entries \( \lambda_1, \lambda_2, \ldots, \lambda_n \), we have
\[
AP = A [\vec{v}_1 \ \vec{v}_2 \ \cdots \ \vec{v}_n] = [A\vec{v}_1 \ A\vec{v}_2 \ \cdots \ A\vec{v}_n],
\]
\[
PD = [\vec{v}_1 \ \vec{v}_2 \ \cdots \ \vec{v}_n] \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = [\lambda_1\vec{v}_1 \ \lambda_2\vec{v}_2 \ \cdots \ \lambda_n\vec{v}_n].
\]
Suppose that $A$ has $n$ linearly independent eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$.

Let $\lambda_i$ be the eigenvalue of $A$ corresponding to $\vec{v}_i$, i.e., $A\vec{v}_i = \lambda_i \vec{v}_i$.

Then $AP = PD$.

Why is $P$ invertible? Because its columns form a linearly independent set, so by the Inverse Matrix Theorem, $P$ is invertible.

Thus we have $D = P^{-1}AP$, and so $A$ is diagonalizable with diagonalizing matrix $P$.

Now suppose that $A$ is diagonalizable.

Then there is an invertible matrix $P$ with columns $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ and a diagonal matrix $D$ with diagonal entries $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that $D = P^{-1}AP$.

So $PD = AP$, which means $A\vec{v}_i = \lambda_i \vec{v}_i$ for each $i = 1, 2, \ldots, n$, that is, each $\vec{v}_i$ is an eigenvector of $A$.

Since $P$ is invertible, the columns of $P$ form an independent set of vectors, and therefore $A$ has $n$ linearly independent eigenvectors.  \hfill \Box

Theorem 6. If an $n \times n$ matrix $A$ has $n$ distinct eigenvalues, then $A$ is diagonalizable.

Proof. Any set of $n$ eigenvectors corresponding to the $n$ distinct eigenvalues are linearly independent, and so $A$ is diagonalizable by Theorem 5.  \hfill \Box

Example. Is $A = \begin{bmatrix} -1 & -3 & -4 \\ 1 & 3 & 2 \\ 1 & 1 & 3 \end{bmatrix}$ diagonalizable?

The characteristic polynomial of $A$ is 

$p(\lambda) = -\lambda^3 + 5\lambda^2 - 8\lambda + 4 = -(\lambda - 1)(\lambda - 2)^2$.

So the eigenvalues of $A$ are $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = 2$.

Row reduction of $A - I$ gives the eigenspace of $A$ belonging the eigenvalue 1 of algebraic multiplicity 1:

$A - I = \begin{bmatrix} -2 & -3 & -4 \\ 1 & 2 & 2 \\ 1 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{Nul}(A - I) = \text{Span} \left( \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right)$.

The geometric multiplicity of the eigenvalue 1 is 1, the dimension of Nul($A - I$).

An eigenvector of $A$ belonging to $\lambda_1 = 1$ is $\vec{v}_1 = [-2 \ 0 \ 1]^T$.

Row reduction of $A - 2I$ gives the eigenspace of $A$ belonging to eigenvalue 2 of algebraic multiplicity 2:

$A - 2I = \begin{bmatrix} -3 & -3 & -4 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{Nul}(A - 2I) = \text{Span} \left( \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right)$.

The geometric multiplicity of the eigenvalue 2 is not 2 but is 1, the dimension of Nul($A - 2I$).
An eigenvector of $A$ belonging to the eigenvalue $2$ is $\vec{v}_2 = [-1 1 0]^T$.

The two eigenvectors $\vec{v}_1, \vec{v}_2$ are linearly independent.

Is there a third eigenvector $\vec{v}_3$ for which the set of $\vec{v}_1, \vec{v}_2, \vec{v}_3$ is linearly independent?

If there were, then $A\vec{v}_3 = \lambda \vec{v}_3$ for an eigenvalue $\lambda$ of $A$, which would mean that $\vec{v}_3 \in \text{Nul}(A - I)$ or $\vec{v}_3 \in \text{Nul}(A - 2I)$, hence $\vec{v}_3$ would be a nonzero scalar multiple of $\vec{v}_1$ or $\vec{v}_2$.

But then $\vec{v}_1, \vec{v}_2, \vec{v}_3$ would form a linearly dependent set.

So, $A$ has only 2 linearly independent eigenvectors, and is not diagonalizable.

Could an $n \times n$ matrix be diagonalizable when it does not have $n$ distinct eigenvalues?

**Theorem 7.** Let $A$ be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \ldots, \lambda_p$.

a. For each $k = 1, \ldots, p$, the geometric multiplicity of $\lambda_k$ is less than or equal to its algebraic multiplicity.

b. The $n \times n$ matrix $A$ is diagonalizable if and only if the sum of the geometric multiplicities of its eigenvalues equals $n$ which happens if and only if the geometric multiplicity of each eigenvalue is equal to its algebraic multiplicity.

c. If $A$ is diagonalizable and $B_k$ is a basis of the eigenspace $\text{Nul}(A - \lambda_k I)$ for each $k$, then the union of the $B_k$ is an eigenvector basis for $\mathbb{R}^n$.

**Example.** Is $A = \begin{bmatrix} 2 & -2 & 2 \\ 0 & 0 & 2 \\ 0 & -1 & 3 \end{bmatrix}$ diagonalizable?

The characteristic polynomial of $A$ is $\det(A - \lambda I) = -(\lambda - 1)(\lambda - 2)^2$.

The eigenvalue $\lambda = 1$ has algebraic multiplicity 1.

Its geometric multiplicity of 1. Why? Because there is a linearly independent solution of $(A - I)x = 0$, but no more than one.

The eigenvalue $\lambda = 2$ has algebraic multiplicity 2.

What is its geometric multiplicity?

We row reduce $A - 2I$ to find out:

$$A - 2I = \begin{bmatrix} 0 & -2 & 2 \\ 0 & -2 & 2 \\ 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. $$

There are two free variables, and so $\dim \text{Nul}(A - 2I) = 2$, meaning the geometric multiplicity of $\lambda = 2$ is 2.

Thus the matrix $A$ is diagonalizable.