Math 313 Lecture #25
§6.1: Inner Product, Length, and Orthogonality

We will now impose on the vector space $\mathbb{R}^n$ a structure that enables us to define geometric notions of length and angle between vectors.

We define the **inner product** of two vectors $\vec{u}, \vec{v}$ in $\mathbb{R}^n$ to be the scalar quantity

$$\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v} = [u_1 \ u_2 \ \ldots \ u_n] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1v_1 + u_2v_2 + \cdots + u_nv_n = \sum_{i=1}^{n} u_iv_i.$$  

This inner product is also known as the **dot product** in $\mathbb{R}^n$.

**Theorem 1.** The inner product $\vec{u} \cdot \vec{v}$ on $\mathbb{R}^n$ satisfies

a. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$,

b. $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$,

c. $(c\vec{u}) \cdot \vec{v} = c(\vec{u} \cdot \vec{v})$, and

d. $\vec{u} \cdot \vec{u} \geq 0$ and $\vec{u} \cdot \vec{u} = 0$ if and only if $\vec{u} = 0$.

The **length**, or **norm** of a vector $\vec{x}$ in $\mathbb{R}^n$ is the nonnegative quantity

$$\| \vec{u} \| = \sqrt{(\vec{u} \cdot \vec{u})} = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}.$$  

An important variation of this is the equation

$$\| \vec{u} \|^2 = \vec{u} \cdot \vec{u}.$$  

For any scalar $c$, the length of $c\vec{u}$ is

$$\| c\vec{u} \| = \sqrt{(cu_1)^2 + (cu_2)^2 + \cdots + (cu_n)^2} = |c| \| \vec{u} \|.$$  

A vector whose length is one is called a **unit vector**.

We always get a unit vector from a nonzero vector when we multiply the vector by the reciprocal of its length:

$$\left\| \frac{1}{\| \vec{u} \|} \vec{u} \right\| = \frac{1}{\| \vec{u} \|} \| \vec{u} \| = 1.$$  

The **distance** between two vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$ is the quantity

$$\| \vec{u} - \vec{v} \| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \cdots + (u_n - v_n)^2}.$$  

Two vectors $\vec{u}$ and $\vec{v}$ form an angle $0 \leq \theta \leq \pi$. How do we measure $\theta$?

**Theorem.** The angle $\theta$ between two vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$ satisfies

$$\vec{u} \cdot \vec{v} = \| \vec{u} \| \| \vec{v} \| \cos \theta.$$
Proof. The vectors \( \vec{u} \), \( \vec{v} \), and \( \vec{v} - \vec{u} \) form the sides a triangle that lies in the subspace \( \text{Span}\{\vec{u}, \vec{v}\} \) (a plane, a line, or a point, all of which lie in a plane).

The law of cosines applies to this triangle:

\[
\|\vec{v} - \vec{u}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\|\cos \theta.
\]

It follows that

\[
\|\vec{u}\|\|\vec{v}\|\cos \theta = \frac{1}{2}(\|\vec{u}\|^2 + \|\vec{v}\|^2 - \|\vec{v} - \vec{u}\|^2)
\]

\[
= \frac{1}{2}(\vec{u} \cdot \vec{u} + \vec{v} \cdot \vec{v} - (\vec{v} - \vec{u}) \cdot (\vec{v} - \vec{u}))
\]

\[
= \frac{1}{2}(\vec{u} \cdot \vec{u} + \vec{v} \cdot \vec{v} - \vec{v} \cdot \vec{v} + \vec{v} \cdot \vec{u} + \vec{u} \cdot \vec{v} - \vec{u} \cdot \vec{u})
\]

\[
= \frac{1}{2}(\vec{v} \cdot \vec{u} + \vec{u} \cdot \vec{v})
\]

\[
= \vec{u} \cdot \vec{v}.
\]

This gives the formula that \( \theta \) satisfies. \( \square \)

Definition. Two vectors \( \vec{u} \) and \( \vec{v} \) in \( \mathbb{R}^n \) are said to be **orthogonal** if the angle between them is \( \pi/2 \), which is to say that their inner product is 0, and we write \( \vec{u} \perp \vec{v} \).

Remember the Pythagorean Theorem that says in a right-angle triangle, the square of the hypotenuse equals the sum of the squares of the other sides?

Theorem 2 (The Pythagorean Theorem). Two vectors \( \vec{u} \) and \( \vec{v} \) in \( \mathbb{R}^n \) are orthogonal if and only if \( \|\vec{u} + \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 \).

Proof. For two vectors \( \vec{u} \) and \( \vec{v} \) in \( \mathbb{R}^n \), we have

\[
\|\vec{u} + \vec{v}\|^2 = (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v}) = \vec{u} \cdot \vec{u} + 2\vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{v} = \|\vec{u}\|^2 + 2\vec{u} \cdot \vec{v} + \|\vec{v}\|^2.
\]

By this identity, if \( \vec{u} \perp \vec{v} \), then \( \vec{u} \cdot \vec{v} = 0 \), and so \( \|\vec{u} + \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 \).

On the other hand, by the identity, if \( \|\vec{u} + \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 \), then \( \vec{u} \cdot \vec{v} = 0 \), and so \( \vec{u} \perp \vec{v} \). \( \square \)

Example. In \( \mathbb{R}^4 \), let \( \vec{x} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 4 \end{bmatrix} \), \( \vec{y} = \begin{bmatrix} -2 \\ 3 \\ 8 \\ 1 \end{bmatrix} \), and \( \vec{z} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \).

Although we can not sketch these vectors, we can use the inner product to get useful information about them.

The inner products of pairs of these vectors are

\[
\vec{x} \cdot \vec{y} = -2 + 6 - 8 + 4 = 0,
\]

\[
\vec{x} \cdot \vec{z} = 1 + 2 + 1 + 4 = 8,
\]

\[
\vec{y} \cdot \vec{z} = -2 + 3 - 8 + 1 = -6.
\]
So \( \vec{x} \perp \vec{y} \), while \( \vec{x} \) is not orthogonal to \( \vec{z} \), and \( \vec{y} \) is not orthogonal to \( \vec{z} \).

The lengths of these vectors are

\[
\| \vec{x} \| = \sqrt{1 + 4 + 1 + 16} = \sqrt{22},
\]
\[
\| \vec{y} \| = \sqrt{4 + 9 + 64 + 1} = \sqrt{78},
\]
\[
\| \vec{z} \| = \sqrt{1 + 1 + 1 + 1} = 2.
\]

So \( \vec{z} \) is the shortest of these three, while \( \vec{y} \) is the longest.

Unit vectors in the direction of \( \vec{x} \), \( \vec{y} \), and \( \vec{z} \) respectively are

\[
\frac{1}{\| \vec{x} \|} \vec{x} = \begin{bmatrix} 1/\sqrt{22} \\ 2/\sqrt{22} \\ -1/\sqrt{22} \\ 4/\sqrt{22} \end{bmatrix}, \quad \frac{1}{\| \vec{y} \|} \vec{y} = \begin{bmatrix} -2/\sqrt{78} \\ 3/\sqrt{78} \\ 8/\sqrt{78} \\ 1/\sqrt{78} \end{bmatrix}, \quad \frac{1}{\| \vec{z} \|} \vec{z} = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ 1/2 \end{bmatrix}.
\]

Now since \( \vec{x} \perp \vec{y} \), the Pythagorean Law should hold for \( \vec{x} \) and \( \vec{y} \). We can verify this:

\[
\| \vec{x} + \vec{y} \|^2 = \| [-1, 5, 7, 5]^T \|^2 = 1 + 25 + 49 + 25 = 100
\]

while

\[
\| \vec{x} \|^2 + \| \vec{y} \|^2 = 22 + 78 = 100.
\]

Orthogonal Complements. We extend the notion of orthogonality between two vectors to between a vector and a subspace of vectors.

A vector \( \vec{z} \) in \( \mathbb{R}^n \) is orthogonal to a subspace \( W \) of \( \mathbb{R}^n \) if \( \vec{z} \) is orthogonal to every vector of \( W \).

The orthogonal complement of \( W \), denoted by \( W^\perp \), is the set of all vectors \( \vec{z} \) in \( \mathbb{R}^n \) that are orthogonal to \( W \).

**Example.** Every line \( L \) through the origin in \( \mathbb{R}^3 \) is a one dimensional subspace.

The orthogonal complement of \( L \) is a plane \( W \) through the origin that makes a right angle with \( L \).

The orthogonal complement of a plane \( W \) through the origin is a line \( L \) through the origin that makes a right angle with \( W \).

**Theorem.** Let \( W \) be a subspace of \( \mathbb{R}^n \). A vector \( \vec{x} \) in \( \mathbb{R}^n \) is orthogonal to \( W \) if and only if \( \vec{x} \) is orthogonal to every vector in a spanning set for \( W \). The orthogonal complement \( W^\perp \) is a subspace of \( \mathbb{R}^n \).

You will provide a proof of this theorem in the homework.

An \( m \times n \) matrix \( A \) has several subspaces associated to it: its null space, its row space, and its column space.

Are any of these the orthogonal complement of others?
Theorem 3. For an \( m \times n \) matrix \( A \), there holds

\[
(\text{Row } A)^\perp = \text{Nul}(A), \text{ and } (\text{Col } A)^\perp = \text{Nul}(A^T).
\]

Proof. By the row-column rule for computing \( A\vec{x} \), we see that for each \( \vec{x} \) in \( \text{Nul}(A) \) is orthogonal to each row of \( A \).

Since the rows of \( A \) are a spanning set for \( \text{Row}(A) \), we have that every \( \vec{x} \) in \( \text{Nul}(A) \) is orthogonal to \( \text{Row}(A) \).

Conversely, if \( \vec{x} \) is orthogonal to \( \text{Row}(A) \), then \( \vec{x} \) is orthogonal to each row of \( A \), and hence \( A\vec{x} = 0 \).

Hence \( \vec{x} \) is in \( \text{Nul}(A) \).

This shows that \( (\text{Row } A)^\perp = \text{Nul}(A) \).

This works for any matrix, so it work for \( A^T \) which gives \( (\text{Row } A^T)^\perp = \text{Nul}(A^T) \).

Since \( \text{Row}(A^T) = \text{Col}(A) \), we have that \( (\text{Col } A)^\perp = \text{Nul}(A^T) \). \( \square \)

Example. Let \( A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \).

The null space of \( A \) is a one dimensional subspace of \( \mathbb{R}^3 \):

\[
\text{Nul}(A) = \text{Span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}.
\]

The row space of \( A \) is a two dimensional subspace of \( \mathbb{R}^3 \):

\[
\text{Row}(A) = \text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \right\}.
\]

Notice that the basis vector for \( \text{Nul}(A) \) is orthogonal to each basis vector for \( \text{Row}(A) \), so indeed \( (\text{Row } A)^\perp = \text{Nul}(A) \).

The null space of \( A^T \) is the zero dimensional subspace of \( \mathbb{R}^2 \):

\[
\text{Nul}(A^T) = \{ \vec{0} \}.
\]

The column space of \( A \) is the two dimensional subspace of \( \mathbb{R}^2 \):

\[
\text{Col}(A) = \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right\}.
\]

Notice that the zero vector is orthogonal to every vector in \( \text{Col}(A) \), so indeed \( (\text{Col } A)^\perp = \text{Nul}(A^T) \).