Math 344 Lecture #11
2.6 Important Example: Bernstein Polynomials

We define and prove that the Bernstein polynomials form a basis for \( F[x; n] \), and relate this basis to the standard basis polynomials. The Bernstein polynomials are used in computer aided design.

Definition 2.6.1. For \( n \in \mathbb{N} \) the Bernstein polynomials \( \{B^n_j\}_{j=0}^n \) of degree \( n \) are defined as

\[
B^n_j(x) = \binom{n}{j} x^j (1 - x)^{n-j},
\]

where

\[
\binom{n}{j} = \frac{n!}{j!(n-j)!}.
\]

Note that each \( B^n_j(x) \) is a polynomial of degree \( n \).

Remark 2.6.2. The values of the Bernstein polynomials \( B^n_j \) at the endpoints of the interval \([0, 1]\) are

\[
B^n_0(0) = 1, \quad B^n_j(0) = 0 \quad \text{for} \quad j \neq 0,
\]

and

\[
B^n_n(1) = 1, \quad B^n_j(1) = 0 \quad \text{for} \quad j \neq n.
\]

Proposition. The sum of the Bernstein polynomials \( \{B^n_j\}_{j=0}^n \) is the constant 1 function, i.e., they form what is called a partition of unity.

Proof. Recall the Binomial Theorem:

\[
(a + b)^n = \sum_{j=0}^n \binom{n}{j} a^j b^{n-j}.
\]

Applying this to \( a = x \) and \( b = 1 - x \) we get

\[
1 = 1^n = (x + (1 - x))^n = \sum_{j=0}^n \binom{n}{j} x^j (1 - x)^{n-j} = \sum_{j=0}^n B^n_j(x)
\]

for all \( x \in F \). \qed

Example 2.6.3. The Bernstein polynomials of degree 4 are

\[
B^4_0(x) = \binom{4}{0} x^0 (1-x)^4 = (1-x)^4 = 1 - 4x + 6x^2 - 4x^3 + x^4,
\]

\[
B^4_1(x) = \binom{4}{1} x^1 (1-x)^3 = 4x(1-x)^3 = 4x - 12x^2 + 12x^3 - 4x^4,
\]

\[
B^4_2(x) = \binom{4}{2} x^2 (1-x)^2 = 6x^2 (1-x)^2 = 6x^2 - 12x^3 + 6x^4,
\]

\[
B^4_3(x) = \binom{4}{3} x^3 (1-x)^1 = 4x^3 (1-x) = 4x^3 - 4x^4,
\]

\[
B^4_4(x) = \binom{4}{4} x^4 (1-x)^0 = x^4.
\]
For \( T = [B_0^4, B_1^4, B_2^4, B_3^4, B_4^4] \) and \( S = [1, x, x^2, x^3, x^4] \), we compute the transition matrix from \( T \) to \( S \):

\[
P_{ST} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
-4 & 4 & 0 & 0 & 0 \\
6 & -12 & 6 & 0 & 0 \\
-4 & 12 & -12 & 4 & 0 \\
1 & -4 & 6 & -4 & 1
\end{bmatrix}.
\]

The transition matrix from \( S \) to \( T \) is

\[
Q_{TS} = P_{ST}^{-1} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & 1/4 & 0 & 0 & 0 \\
1 & 1/2 & 1/6 & 0 & 0 \\
1 & 3/4 & 1/2 & 1/4 & 0 \\
1 & 1 & 1 & 1 & 1
\end{bmatrix}.
\]

For \( p(x) = 1 + 8x - 12x^2 - 4x^3 + 5x^4 \) we find the coordinates of \( p(x) \) in the basis \( T \) by computing

\[
[p(x)]_T = Q_{TS}[p(x)]_S = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & 1/4 & 0 & 0 & 0 \\
1 & 1/2 & 1/6 & 0 & 0 \\
1 & 3/4 & 1/2 & 1/4 & 0 \\
1 & 1 & 1 & 1 & 1
\end{bmatrix} \begin{bmatrix}1 \\ 8 \\ -12 \\ -4 \\ 5 \end{bmatrix} = \begin{bmatrix}1 \\ 3 \\ 3 \\ 0 \\ -2 \end{bmatrix}.
\]

Thus we have

\[
p(x) = B_0^n(x) + 3B_1^n(x) + 3B_2^n(x) + 0B_3^n(x) - 2B_4^n(x).
\]

To prove that the Bernstein polynomials of degree \( n \) form a basis for \( F[x; n] \) we need the following result, which expresses each Bernstein polynomial in terms of the standard basis.

**Lemma 2.6.4.** For \( n \in \mathbb{N} \), the Bernstein polynomials \( \{B_j^n\}_{j=0}^n \) satisfy

\[
B_j^n(x) = \sum_{i=j}^{n} (-1)^{i-j} \binom{n}{i} \binom{i}{j} x^i.
\]

See the Appendix for a proof.

**Theorem 2.6.5.** For each \( n \in \mathbb{N} \), the Bernstein polynomials \( \{B_j^n\}_{j=0}^n \) of degree \( n \) form a basis for \( F[x; n] \).

See the Appendix for a proof.

**Example.** In the standard basis \( S = [1, x, x^2, x^3, x^4] \) of \( F[x; 4] \) the linear operator \( L \) of differentiation has the matrix representation

\[
A_{SS} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 3 & 0 \\
0 & 0 & 0 & 0 & 4 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]
The matrix representation of $L$ in the Bernstein basis $T = [B^1_0, B^1_1, B^1_2, B^4_3, B^4_4]$ is given by the similarity

$$B_{TT} = P_{ST}^{-1} A_{SS} P_{ST} = \begin{bmatrix}
-4 & 4 & 0 & 0 & 0 \\
-1 & -2 & 3 & 0 & 0 \\
0 & -2 & 0 & 2 & 0 \\
0 & 0 & -3 & 2 & 1 \\
0 & 0 & 0 & -4 & 4
\end{bmatrix}.$$

This a “tri-diagonal” matrix, i.e., zeroes in the entries outside of the main diagonal, the super diagonal, and the sup diagonal.

Notice how the ordering of the bases makes sense: $T$ to $S$ by $P_{ST}$, then $S$ to $S$ by $A_{SS}$, and finally $S$ to $T$ by $P_{ST}^{-1}$ (keeping in mind that the inverse switches the ordering of the bases).

Recall that the coordinates of $p(x) = 1 + 8x - 12x^2 - 4x^3 + 5x^4$ are given by

$$[p(x)]_T = \begin{bmatrix}
1 \\
3 \\
3 \\
0 \\
-2
\end{bmatrix}.$$

So the derivative of $p(x)$ in terms of $T$ is

$$B_{TT}[p(x)]_T = \begin{bmatrix}
-4 & 4 & 0 & 0 & 0 \\
-1 & -2 & 3 & 0 & 0 \\
0 & -2 & 0 & 2 & 0 \\
0 & 0 & -3 & 2 & 1 \\
0 & 0 & 0 & -4 & 4
\end{bmatrix} \begin{bmatrix}
1 \\
3 \\
3 \\
0 \\
-2
\end{bmatrix} = \begin{bmatrix}
8 \\
2 \\
3 \\
0 \\
-2
\end{bmatrix}.$$

Thus

$$p'(x) = 8B^4_0(x) + 2B^4_2(x) - 6B^4_2(x) - 11B^4_3(x) - 8B^4_4(x).$$

**Example.** The Bernstein polynomials of different degrees are related.

For example, we have

$$B^4_2(x) = \binom{4}{2} x^2(1 - x)^{4-2} = 6x^2 - 12x^3 + 6x^4$$

and

$$(1 - x)B^3_2(x) + xB^3_1(x) = (1 - x)(3x^2(1 - x)) + x(3x(1 - x)^2)$$

$$= 3x^2(1 - x)^2 + 3x^2(1 - x)^2$$

$$= 6x^2(1 - x)^2$$

$$= \binom{4}{2} x^2(1 - x)^{4-2}$$

$$= B^4_2(x).$$

You have two HW (Exercises 2.31 and 2.32) to prove general relationships like this.
Appendix

Proof of Lemma 2.6.4. Fix \( j \in \{0, 1, \ldots, n\} \).

By the Binomial Theorem we have

\[
(1 - x)^{n-j} = \sum_{k=0}^{n-j} \binom{n-j}{k} (-1)^k x^k.
\]

Thus

\[
B^n_j(x) = \binom{n}{j} x^j (1 - x)^{n-j} = \binom{n}{j} x^j \sum_{k=0}^{n-j} \binom{n-j}{k} (-1)^k x^k.
\]

The two terms on the left of the sum on the right-hand side can be moved inside the sum to give

\[
\sum_{k=0}^{n-j} \binom{n}{j} \binom{n-j}{k} (-1)^k x^{j+k}.
\]

Replacing \( k \) with \( k = i - j \) where \( i = j, \ldots, n \) we get

\[
\sum_{i=j}^{n} (-1)^{i-j} \binom{n}{j} \binom{n-j}{i-j} x^i.
\]

We simplify

\[
\binom{n}{j} \binom{n-j}{i-j} = \frac{n!}{j!(n-j)!(i-j)!(n-j-i+j)!} = \frac{(n-j)!}{j!(i-j)!(n-i)!(n-j-i+j)!} = \frac{n!}{j!(n-i)!i!} = \frac{i!}{i!(n-i)!j!(i-j)!} = \binom{n}{i} \binom{i}{j}
\]

We arrive at

\[
B^n_j(x) = \sum_{i=j}^{n} (-1)^{i-j} \binom{n}{i} \binom{i}{j} x^i.
\]

This gives the result. \( \square \)

Proof of Theorem 2.6.5. Let \( T = [B^n_0(x), B^n_1(x), \ldots, B^n_n(x)] \).

We will show that these \( n+1 \) polynomials are linearly independent in \( \mathbb{F}[x; n] \), thus making \( T \) a basis.
We construct (what will be transition matrices) $P$ and $Q$ such that $Q = P^{-1}$ and such that any linear equation

$$\sum_{i=0}^{n} c_j B^n_j(x) = 0$$

can be written in terms of the standard basis $S = [1, x, \ldots, x^n]$ of $\mathbb{F}[x; n]$ as

$$\begin{bmatrix} 1 & x & \cdots & x^n \end{bmatrix} P \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Such an equation implies that $P \begin{bmatrix} c_0 & c_1 & \cdots & c_n \end{bmatrix}^T = 0$ since $S$ is a basis.

Since $QP = I$ we have

$$\begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix} = (QP) \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix} = Q \left( P \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix} \right) = Q \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

This implies that $T$ is linearly independent.

So it remains to construct $P$ and $Q$.

By Lemma 2.6.4, we know that

$$B^n_j(x) = \sum_{k=j}^{n} (-1)^{j-k} \binom{n}{j} \binom{j}{k} x^k$$

for each of $j = 0, 1, \ldots, n$. This expresses each Bernstein polynomial in terms of $S$.

Let $P = [p_{jk}]$ be the $(n + 1) \times (n + 1)$ lower triangle matrix defined by

$$p_{jk} = \begin{cases} (-1)^{j-k} \binom{n}{j} \binom{j}{k} & \text{if } j \geq k, \\ 0 & \text{if } j < k, \end{cases}$$

where $j, k \in \{0, 1, \ldots, n\}$.

We show that $P$ is invertible by direct computation with a candidate matrix $Q$ for the inverse.

We define the $(n + 1) \times (n + 1)$ matrix $Q = [q_{ij}]$ by

$$q_{ij} = \begin{cases} \binom{i}{j} / \binom{n}{j} & \text{if } i \geq j, \\ 0 & \text{if } i < j, \end{cases}$$
for \( i, j \in \{0, 1, \ldots, n\} \).

When \( 0 \leq k \leq i \leq n \), the \( ik \) entry of \( QP \) is

\[
(QP)_{ik} = \sum_{j=0}^{n} q_{ij}p_{jk} = \sum_{j=k}^{i} q_{ij}p_{jk} = \sum_{j=k}^{i} (-1)^{j-k} \binom{i}{j} \binom{j}{k} 1^j = B^i_k(1).
\]

When \( i = k \) we have \( B^i_k(1) = 1 \) and when \( k < i \) we have \( B^i_k(1) = 0 \).

Also when \( 0 \leq i < k \leq n \), the entry \((QP)_{ik} = 0 \) because both matrices are lower triangular.

This gives \( QP = I \). \qed