3.7 Adjoint

3.7.1 Finite Dimensional Riesz Representation Theorem

Let \((V, \langle \cdot, \cdot \rangle)\) be an inner product space, and \(\|x\| = \sqrt{\langle x, x \rangle}\) its associated norm.

**Proposition.** Associated to each \(y \in V\) is a linear transformation \(f : V \to \mathbb{F}\) defined by \(f(x) = \langle y, x \rangle\) for which \(\|f\| = \|y\|\).

**Proof.** That \(f(x) = \langle y, x \rangle\) is a linear transformation follows from the properties of the inner product.

By the Cauchy-Schwarz Inequality we have
\[
|f(x)| = |\langle y, x \rangle| \leq \|y\| \|x\|
\]
which implies that
\[
\|f\| \leq \|y\|.
\]
Since \(|f(x)| \leq \|f\| \|x\|\) for all \(x \in V\), we get for \(x = y\) that
\[
\|y\|^2 = \langle y, y \rangle = f(y) = |f(y)| \leq \|f\| \|y\|,
\]
which implies that \(\|y\| \leq \|f\|\).

Thus \(\|f\| = \|y\|\). \(\square\)

**Theorem 3.7.1 (The Finite Dimensional Riesz Representation Theorem).** For an inner product space \((V, \langle \cdot, \cdot \rangle)\), if \(L : V \to \mathbb{F}\) is a linear transformation, then there is a unique \(y \in V\) such that \(f(x) = \langle y, x \rangle\) for all \(x \in V\).

**Proof.** Let \(S = [x_1, \ldots, x_n]\) be an orthonormal basis for \(V\).

For \(x = \sum_{i=1}^{n} a_i x_i\) we have \(a_i = \langle x_i, x \rangle\) and
\[
L(x) = \sum_{i=1}^{n} a_i f(x_i) = \sum_{i=1}^{n} \langle x_i, x \rangle L(x_i) = \sum_{i=1}^{n} \langle L(x_i) x_i, x \rangle = \sum_{i=1}^{n} \langle L(x_i) x_i, x \rangle.
\]
Thus by setting
\[
y = \sum_{i=1}^{n} \overline{L(x_i)} x_i
\]
we get \(L(x) = \langle y, x \rangle\) for all \(x \in V\).

To show that this \(y\) is unique, suppose that \(L(x) = \langle y', x \rangle\) for another \(y \in V\).

Then for all \(x \in V\) there holds
\[
0 = \langle y, x \rangle - \langle y', x \rangle = \langle y - y', x \rangle.
\]
Taking \(x = y - y'\) in this gives
\[
0 = \langle y - y', y - y' \rangle = \|y - y'\|.
\]
Thus \(y' = y\). \(\square\)
Definition 3.7.4. A linear functional on a vector space $V$ over $\mathbb{F}$ is a linear transformation $L : V \to \mathbb{F}$.

For a normed linear space $(V, \| \cdot \|_V)$, a linear functional $L \in \mathcal{L}(V, \mathbb{F})$ is bounded if $\|f\|_V < \infty$.

The normed vector space $\mathcal{B}(V, \mathbb{F})$ is the collection of all bounded linear functionals on $(V, \| \cdot \|)$.

Corollary 3.7.5. If $V$ is a finite dimensional normed linear space then $\mathcal{B}(V, \mathbb{F}) = \mathcal{L}(V, \mathbb{F})$.

3.7.2 Adjoints

Definition 3.7.6. Suppose $(V, \langle \cdot, \cdot \rangle_V)$ and $(W, \langle \cdot, \cdot \rangle_W)$ are inner product spaces. An adjoint of $L \in \mathcal{L}(V, W)$ is an $L^* \in \mathcal{L}(W, V)$ such that

$$\langle w, L(v) \rangle_W = \langle L^*(w), v \rangle_V$$

for all $v \in V, w \in W$.

An adjoint $L^*$ of $L \in \mathcal{L}(V, W)$, if it exists, has the property that for a linear functional $g : W \to \mathbb{F}$ given by $g(z) = \langle w, z \rangle$ for some $w \in W$, the “pull-back” $f = g \circ L : V \to \mathbb{F}$ is a linear functional on $V$ given by

$$f(v) = (g \circ L)(v) = g(L(v)) = \langle w, L(v) \rangle_W = \langle L^*(w), v \rangle_V,$$

i.e., we can express the pull-back as an inner product on $V$ instead of on $W$.

Example 3.7.8. Let $A \in M_{m \times n}(\mathbb{F})$ be the matrix representation of a linear transformation $L : \mathbb{F}^n \to \mathbb{F}^m$ with respect to the standard bases. For the standard inner products on $\mathbb{F}^n$ and $\mathbb{F}^m$, an adjoint $L^*$ of $L$ is represented by the Hermitian conjugate (or conjugate transpose) $A^\dagger$ because

$$\langle y, Ax \rangle = y^H Ax = (A^H y)^H x = \langle A^H y, x \rangle.$$

Example 3.7.9. Let $V$ be the vector space of infinitely differentiable functions $f : \mathbb{R} \to \mathbb{R}$ that have compact support, i.e., to each $f$ there is a compact interval $[a, b]$ such that $f(x) = 0$ for $x \in \mathbb{R} - [a, b]$. Each of these functions is Riemann integrable having

$$\int_{-\infty}^{\infty} |f(x)| \, dx < \infty.$$

The space $V$ is an inner product space with the inner product

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x) g(x) \, dx.$$

Let $L \in \mathcal{L}(V)$ be defined by $L[g](x) = g'(x)$. An adjoint of $L$ is $L^* = -L$ because, by integration by parts and the compact support, we have

$$\langle f, L[g] \rangle = \int_{-\infty}^{\infty} f(x) g'(x) \, dx = -\int_{-\infty}^{\infty} f'(x) g(x) \, dx = \langle -L[f], g \rangle.$$
Theorem 3.7.10. Let \((V, \langle \cdot , \cdot \rangle_V)\) and \((W, \langle \cdot , \cdot \rangle_W)\) be finite dimensional inner product spaces over the same field \(\mathbb{F}\). For each \(L \in \mathcal{L}(V,W)\) there exists a unique adjoint \(L^* \in \mathcal{L}(W,V)\).

Proof. First we prove existence of \(L^* \in \mathcal{L}(W,V)\) for a \(L \in \mathcal{L}(V,W)\).

For each \(w \in W\) define the transformation \(L_w : V \to \mathbb{F}\) by \(L_w(v) = \langle v, L(v) \rangle_W\).

Since \(L\) is linear, and the inner product is linear in the second slot, we have \(L_w \in \mathcal{L}(V, \mathbb{F})\).

By Theorem 3.7.1 (the finite dimensional version of the Riesz Representation Theorem), there is associated to each \(w \in W\) a unique \(u \in V\) such that \(L_w(v) = \langle u, v \rangle_V\).

We define a map \(L^* : W \to V\) by \(L^*(w) = u\) which satisfies

\[
\langle w, L(v) \rangle_W = L_w(v) = \langle u, v \rangle_V = \langle L^*(w), v \rangle_V \quad \text{for all } v \in V, w \in W.
\]

To show that \(L^*\) is linear, we have for any \(w_1, w_2 \in W\) and any \(a, b \in \mathbb{F}\) that

\[
\langle L^*(aw_1 + bw_2), v \rangle_V = \langle aw_1 + bw_2, L(v) \rangle_W
= \overline{a} \langle w_1, L(v) \rangle_W + \overline{b} \langle w_2, L(v) \rangle_W
= \overline{a} \langle L^*(w_1), v \rangle_V + \overline{b} \langle L^*(w_2), v \rangle_V
= \langle aL^*(w_1) + bL^*(w_2), v \rangle_V,
\]

which shows that \(L^*\) is linear, and hence \(L^* \in \mathcal{L}(W,V)\).

To show that \(L^*\) is unique, suppose \(L_1^*\) and \(L_2^*\) are adjoints for \(L\). Then

\[
\langle L_1^*(w), v \rangle_W = \langle w, L(v) \rangle_V = \langle L_2^*(w), v \rangle_V,
\]

for all \(w \in W\) and all \(v \in V\), so that

\[
0 = \langle L_2^*(w), v \rangle_V - \langle L_1^*(w), v \rangle_W = \langle (L_2^* - L_1^*)(w), v \rangle_V
\]

for all \(w \in W\) and all \(v \in V\).

By Proposition 3.1.16 (if \(\langle x, y \rangle = 0\) for all \(y\), then \(x = 0\)), we get \((L_1^* - L_2^*)(w) = 0\) for all \(w \in W\).

This implies that \(L_2^*(w) = L_1^*(w)\) for all \(w \in W\), whence that \(L_2^* = L_1^*\). \(\square\)

Remark 3.7.7. Since the adjoint exists and is unique for finite dimensional inner product spaces, we speak of the adjoint of a linear transformation in this situation.

Proposition 3.7.12. Let \(V\) and \(W\) be finite-dimensional inner product spaces over the same field \(\mathbb{F}\).

(i) If \(S, T \in \mathcal{L}(V, W)\), then \((S + T)^* = S^* + T^*\) and \((\alpha T)^* = \pi T^*\) for \(\alpha \in \mathbb{F}\).
(ii) If \(S \in \mathcal{L}(V, W)\), then \((S^*)^* = S\).
(iii) If \(S, T \in \mathcal{L}(V)\), then \((ST)^* = T^* S^*\).
(iv) If \(T \in \mathcal{L}(V)\) is invertible, then \((T^*)^{-1} = (T^{-1})^*\).

The proof of this is HW (Exercise 3.39).