11.7 The Residue Theorem

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As always \((X, \| \cdot \|)\) is a complex Banach space.

Here is an outline for today.

- Isolated Singularities
- Residues and Winding Numbers
- The Residue Theorem

First Reading Quiz Question:

- What are the three types of isolated singularities that a holomorphic function can have?
- How is the residue of a holomorphic function at an isolated singularity computed?
Definition 11.7.1. For a point $z_0 \in \mathbb{C}$, an $\epsilon > 0$, and the punctured open disk

$$U = \{ z \in \mathbb{C} : 0 < |z - z_0| < \epsilon \},$$

for $f : U \to X$ holomorphic, we say that $z_0$ is an isolated singularity of $f$ if $f$ is not assumed complex differentiable at $z_0$.

For an isolated singularity $z_0$ of $f$ the principal part of the Laurent series

$$\sum_{k=-\infty}^{\infty} a_k(z - z_0)^k$$

of $f$ on $B(z_0, \epsilon) \setminus \{z_0\}$ is the series

$$\sum_{k=-\infty}^{-1} a_k(z - z_0)^k.$$

We use the principal part to classify isolated singularities.
Definition. An isolated singularity \( z_0 \) of \( f \) is called a **removable singularity** if the principal part of the Laurent series of \( f \) about \( z_0 \) is zero, i.e., \( a_k = 0 \) for all \( k = -1, -2, -3, \ldots \).

If \( f \) has a removable singularity at \( z_0 \), then \( f \) extends to a holomorphic function on \( B(z_0, \epsilon) \) by means of the power series \( \sum_{k=0}^{\infty} a_k(z - z_0)^k \) convergent on \( B(z_0, \epsilon) \).

Example. The function

\[
f(z) = \frac{\cos(z) - 1}{z^2} = \frac{1}{z^2} \sum_{k=1}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!} = \sum_{k=1}^{\infty} \frac{(-1)^k z^{2k-2}}{(2k)!}
\]

\[
= -\frac{1}{2} + \frac{z^2}{4!} + \cdots
\]

has a removable singular at the isolated singularity \( z_0 = 0 \) of \( f \).
Definition. An isolated singular $z_0$ of $f$ is called a **pole of order** $N \in \mathbb{N}$ if the principal part of the Laurent series of $f$ about $z_0$ has the form

$$f(z) = \sum_{k=-N}^{-1} a_k (z - z_0)^k,$$

i.e., $a_k = 0$ for all $k < -N$ in the Laurent series for $f$ about $z_0$.

A pole of order 1 is called a **simple pole**.

Example. The function

$$f(z) = \frac{\sin(z)}{z^4} = \frac{1}{z^4} \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!} = \frac{1}{z^3} - \frac{1}{6z} + \frac{z}{5!} + \cdots$$

has a pole of order $N = 3$ at the isolated singularity $z_0 = 0$ of $f$. 
Definition. An isolated singularity $z_0$ of $f$ is called an essential singularity if the principal part of the Laurent series for $f$ about $z_0$ has infinitely many nonzero terms, i.e., $a_k \neq 0$ for infinitely many $-k \in \mathbb{N}$.

Example. The function

$$f(z) = \sin \left( \frac{1}{z} \right) = \sum_{k=0}^{\infty} \frac{(-1)^k (1/z)^{2k+1}}{(2k + 1)!}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k z^{-2k-1}}{(2k + 1)!}$$

$$= \frac{1}{z} - \frac{1}{6z^3} + \frac{1}{5!z^5} - \cdots$$

has an essential singularity at the isolated singularity $z_0 = 0$ of $f$. 

What questions do you have?
Definition 11.7.4. For an open set $U$ in $\mathbb{C}$ and finitely many distinct points $z_1, \ldots, z_n$ in $U$, a function

$$f : U \setminus \{z_1, \ldots, z_n\} \to X$$

is called meromorphic if $f$ is holomorphic on the open set $U \setminus \{z_1, \ldots, z_n\}$ with $f$ having poles at each $z_i$.

Example 11.7.5. For polynomials $p$ and $q$ with $q$ not identically equal to 0, the rational function

$$p(z)/q(z),$$

in lowest terms (i.e., any common factors that $p$ and $q$ have have already been cancelled), is a meromorphic function on $\mathbb{C} \setminus \{z_1, \ldots, z_k\}$ where $z_1, \ldots, z_k$ are the distinct roots of $q$.

FYI: It is standard practice is always assume that a rational function is given in lowest terms, unless explicitly told otherwise.
Remark. We have already seen that the coefficient $a_{-1}$ of the power $(z - z_0)^{-1}$ in the Laurent series of a function $f$ holomorphic on a punctured disk $B(z_0, \epsilon) \setminus \{z_0\}$ is the quantity needed when computing contour integrals of $f$ on simply closed curves with $z_0$ in its interior.

Because of the importance of this coefficient, we give it a name.

Definition 11.7.6. For a holomorphic $f : B(z_0, \epsilon) \setminus \{z_0\} \to X$ and simple close curve $\gamma$ in $B(z_0, \epsilon) \setminus \{z_0\}$, the quantity

$$\frac{1}{2\pi i} \oint_{\gamma} f(z) \, dz$$

is called the residue of $f$ at $z_0$ and is denoted by $\text{Res}(f, z_0)$.

Proposition 11.7.7. If $f : B(z_0, \epsilon) \setminus \{z_0\} \to X$ is holomorphic, then $\text{Res}(f, z_0)$ is the coefficient $a_{-1}$ of the power $(z - z_0)^{-1}$ in the Laurent series of $f$ about $z_0$. 
Proposition 11.7.8. Suppose a holomorphic $f$ has an isolated singularity at $z_0$.

(i) The isolated singularity at $z_0$ is removable if and only if
$$\lim_{z \to z_0} f(z)$$
exists (as a complex number; the book inaccurately uses the term finite).

(ii) If for some nonnegative integer $k$ the limit
$$\lim_{z \to z_0} (z - z_0)^k f(z)$$
exists (as a complex number), then the isolated singularity $z_0$ of $f$ is either a removable singularity or a pole of order equal to or less than $k$.

(iii) If the limit $\lim_{z \to z_0} (z - z_0)f(z)$ exists (as a complex number), then
$$\text{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0)f(z).$$

The proof of this is HW (Exercise 11.29).
What questions do you have?
Journey to the Residue Theorem

Consider the contour integral

\[
\frac{1}{2\pi i} \oint_\gamma \frac{1}{z - z_0} \, dz
\]

for the closed contour \( \gamma : [0, 2k\pi] \rightarrow \mathbb{C} \) given by \( \gamma(t) = z_0 + e^{i\theta} \) for a positive integer \( k \).

Computing this contour integral gives

\[
\frac{1}{2\pi i} \int_0^{2k\pi} \frac{1}{e^{i\theta}(i e^{i\theta})} \, d\theta = \frac{1}{2\pi} \int_0^{2k\pi} d\theta = \frac{2k\pi}{2\pi} = k.
\]

The closed contour \( \gamma \) goes around \( z_0 \) in the counterclockwise direction \( k \) times while the residue of \( 1/(z - z_0) \) at \( z_0 \) is 1.

If this same curve \( \gamma \) is traversed in the clockwise direction, i.e., \( \gamma(\theta) = z_0 + e^{-i\theta} \), then we would get \(-k\) as the value of the contour integral.
Furthermore, if $\gamma$ is closed contour that does not enclose $z_0$, then $1/(z - z_0)$ is holomorphic on a simply connected open set containing $\gamma$ but not containing $z_0$, so that by the Cauchy-Goursat Theorem we have

$$\frac{1}{2\pi i} \oint_{\gamma} \frac{1}{z - z_0} \, dz = 0.$$
These observations motivate the notion of the winding number.

Definition 11.7.9. For a closed contour $\gamma$ in $\mathbb{C}$ and $z_0$ a point of $\mathbb{C}$ not on $\gamma$, the winding number of $\gamma$ with respect to $z_0$ is the quantity

$$I(\gamma, z_0) = \frac{1}{2\pi i} \oint_{\gamma} \frac{1}{z - z_0} \, dz.$$ 

Lemma 11.7.12. For a simply connected open set $U$ in $\mathbb{C}$, a closed contour $\gamma$ in $U$, and a point $z_0 \in U$ not on $\gamma$, if

$$N(z) = \sum_{k=0}^{\infty} \frac{b_k}{(z - z_0)^k}$$

is uniformly convergent on compact subsets of $U \setminus \{z_0\}$, then there holds

$$\frac{1}{2\pi i} \oint_{\gamma} N(z) \, dz = \text{Res}(N, z_0) I(\gamma, z_0).$$

The proof of this is HW (Exercise 11.30).
What questions do you have?
Theorem 11.7.13 (The Residue Theorem). For a simply connected $U$ in $\mathbb{C}$ and finitely many points $z_1, \ldots, z_n \in U$, if 

$$f : U \setminus \{z_1, \ldots, z_n\} \to X$$

is holomorphic and $\gamma$ is a closed contour in $U \setminus \{z_1, \ldots, z_n\}$, then

$$\frac{1}{2\pi i} \oint_{\gamma} f(z) \, dz = \sum_{j=1}^{n} \text{Res}(f, z_j) I(\gamma, z_j).$$

Second Quiz Question: The Cauchy-Goursat Theorem and the Cauchy Integral Formula are special cases of the Residue Theorem.

True
The Residue Theorem has the Cauchy-Goursat Theorem as a special case.

When \( f : U \to X \) is holomorphic, i.e., there are no points in \( U \) at which \( f \) is not complex differentiable, and \( \gamma \) in \( U \) is a simple closed curve, we select any \( z_0 \in U \setminus \gamma \).

The residue of \( f \) at \( z_0 \) is 0 by Proposition 11.7.8 part (iii), i.e.,

\[
\text{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0)f(z) = 0;
\]

hence, regardless of the value of \( I(\gamma, z_0) \), the Residue Theorem gives

\[
\oint_{\gamma} f(z) \, dz = 0.
\]
The Residue Theorem has Cauchy's Integral formula also as special case.

When $f : U \rightarrow X$ is holomorphic, and $z_0 \in U$, then the function $g(z) = \frac{f(z)}{z - z_0}$ is holomorphic on $U \setminus \{z_0\}$, so for any simple closed curve $\gamma$ in $U$ enclosing $z_0$ the Residue Theorem gives

$$
\frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{z - z_0} \, dz = \frac{1}{2\pi i} \oint_{\gamma} g(z) \, dz = \text{Res}(g, z_0) I(\gamma, z_0);
$$

here $I(\gamma, z_0) = 1$ because $\gamma$ is a simple closed curve enclosing $z_0$, and $\text{Res}(g, z_0) = f(z_0)$ because using the power series for $f$ about $z_0$ gives the Laurent series

$$
g(z) = \frac{f(z)}{z - z_0} = \frac{1}{z - z_0} \sum_{k=0}^{\infty} f^{(k)}(z_0) \frac{(z-z_0)^k}{k!} = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z-z_0)^{k-1}
$$

in which the coefficient of $(z - z_0)^{-1}$ is $f(z_0)$. 
What questions do you have?
To use the Residue Theorem requires that we compute the required residues.

We have seen two ways to compute the residue of $f$ at a point $z_0$: by computing the Laurent series of $f$ on $B(z_0, \epsilon) \setminus \{z_0\}$, or by Proposition 11.7.8 part (iii).

Of the many other means of computing $\text{Res}(f, z_0)$ we mention another one.

Proposition 11.7.15. Suppose $g : B(z_0, \epsilon) \to X$ and $h : B(z_0, \epsilon) \to \mathbb{C}$ are holomorphic. If $g(z_0) \neq 0$, $h(z_0) = 0$, and $h'(z_0) \neq 0$, then the function $g(z)/h(z) : B(z_0, \epsilon) \setminus \{z_0\} \to X$ is meromorphic with a simple pole at $z_0$ and

$$\text{Res} \left( \frac{g(z)}{h(z)}, z_0 \right) = \frac{g(z_0)}{h'(z_0)}.$$
Note. While you are responsible for knowing and using Proposition 11.7.15, you are NOT responsible for the next proposition on computing the residue for a pole of order 2.

It is given to show you how complicated residue calculations can become for nonsimple poles.

Proposition. Suppose \( g : B(z_0, \epsilon) \rightarrow X \) and \( h : B(z_0, \epsilon) \rightarrow \mathbb{C} \) are holomorphic. If \( g(z_0) \neq 0, h(z_0) = 0, h'(z_0) = 0, \) and \( h''(z_0) \neq 0, \) then \( g(z)/h(z) : B(z_0, \epsilon) \setminus \{z_0\} \rightarrow X \) is meromorphic with a pole of order 2 at \( z_0, \) and

\[
\text{Res} \left( \frac{g(z)}{h(z)}, z_0 \right) = \frac{2g'(z_0)}{h''(z_0)} - \frac{2g(z_0)h^{(3)}(z_0)}{3[h''(z_0)]^2}.
\]

REMEMBER YOU ARE NOT RESPONSIBLE FOR THIS FORMULA.
Remark. A truly hideous formula for the residue of a pole of order $N$ is given in my lecture notes.

It involves symbolic cofactor expansion of an $N \times N$ matrix.

Need I say more????

Remark. Unfortunately for an essential singularity of $f$ at $z_0$ there are no “simple” formulas for computing the residue of $f$ at $z_0$.

We typically rely on computing, somehow, the Laurent series for $f$ at $z_0$ to find its residue at $z_0$. 
Example (in lieu of 11.7.16). For the holomorphic function

\[ f(z) = 1/(z^2 + 1) \]

the numerator is \( g(z) = 1 \) and the denominator is \( h(z) = z^2 + 1 \).

The roots of \( h(z) = (z - i)(z + i) \) are \( z_1 = i \) and \( z_2 = -i \), i.e., \( h(z_1) = 0 \) and \( h(z_2) = 0 \).

Since \( h'(z) = 2z \) we have

\[ h'(z_1) = 2i \neq 0 \quad \text{and} \quad h'(z_2) = -2i \neq 0. \]

By Proposition 11.7.15, the function \( f \) has a simple pole at each of \( z_1 \) and \( z_2 \) where

\[ \text{Res}(f, z_1) = \frac{g(z_1)}{h'(z_1)} = \frac{1}{2i} \quad \text{and} \quad \text{Res}(f, z_2) = \frac{g(z_2)}{h'(z_2)} = -\frac{1}{2i}. \]
The simple closed contour \( \gamma = \{ z \in \mathbb{C} : |z| = 2 \} \), i.e., the circle centered at 0 with radius 2, encloses both simple poles of \( f \).

[Draw the picture]

For the winding numbers we have \( I(\gamma, z_1) = 1 \) and \( I(\gamma, z_2) = 1 \).

By the Residue Theorem we compute

\[
\frac{1}{2\pi i} \oint_{\gamma} f(z) \, dz = \sum_{j=1}^{2} \text{Res}(f, z_j) I(\gamma, z_j) = \frac{1}{2i} - \frac{1}{2i} = 0.
\]
What questions do you have?
Example 11.7.17. Compute
\[ \int_{-\infty}^{\infty} f(x) \, dx \] for \( f(x) = \frac{1}{1 + x^4} \).

The improper integral of \( f \) over \( \mathbb{R} \) converges by a comparison test with \( 1/(1 + x^2) \), i.e., since \( 1 + x^4 \geq 1 + x^2 \), then
\[ 0 \leq \frac{1}{1 + x^4} \leq \frac{1}{1 + x^2} \]
and the improper integral of \( 1/(1 + x^2) \) converges because
\[ \int_{-\infty}^{\infty} \frac{1}{1 + x^2} \, dx = \lim_{R \to \infty} \arctan(x) \bigg|_{-R}^{R} = \pi < \infty. \]

Convergence of the improper integral of \( 1/(1 + x^4) \) over \( \mathbb{R} \) justifies writing
\[ \int_{-\infty}^{\infty} \frac{1}{1 + x^4} \, dx = \lim_{R \to \infty} \int_{-R}^{R} \frac{1}{1 + x^4} \, dx. \]
We recognize that the integrand is equal to the complex-valued function

\[ f(z) = \frac{1}{1 + z^4} \text{ when } z \in \mathbb{R}. \]

The function \( f(z) \) is complex differentiable except at the four roots of the denominator \( h(z) = 1 + z^4 \).

We can find these four roots using Euler’s Formula as follows.

By writing

\[ -1 = e^{i\pi + 2in\pi} \]

for an arbitrary integer \( n \), the equation

\[ 1 + z^4 = 0 \text{ becomes } e^{i\pi + 2in\pi} = z^4. \]

Taking fourth roots of both sides of this equation gives

\[ e^{i\pi/4 + ni\pi/2} = z. \]
The root complex roots of \( h(z) = z^4 + 1 \) are correspond to the four distinct angles \( \pi/4, 3\pi/4, 5\pi/4, 7\pi/4 \) in \([0, 2\pi)\); the four roots are

\[
z_1 = e^{i\pi/4}, z_2 = e^{3i\pi/4}, z_3 = e^{5i\pi/4}, z_4 = e^{7i\pi/4}.
\]

There is one root in each quadrant of the complex plane.

The function \( f \) is meromorphic on \( \mathbb{C} \setminus \{z_1, z_2, z_3, z_4\} \).

Since \( h'(z) = 4z^3 \) and \( h'(z_j) \neq 0 \) for all \( j = 1, 2, 3, 4 \), each point \( z_j \) is a simple pole for \( f(z) = 1/h(z) \) with residue

\[
\text{Res}(f, z_j) = \frac{1}{h'(z_j)} = \frac{1}{4z_j^3}.
\]

Now for the “magic” of the Residue Theorem.
For $R \geq 2$, form the closed simple contour $D$ that is the sum of the line $\gamma$ from $-R$ to $R$ and the top half $C$ of the circle with center 0 and radius $R$ traversed counterclockwise. [Draw the picture]

This gives

$$\oint_D f(z)\,dz = \int_\gamma f(z)\,dz + \int_C f(z)\,dz = \int_{-R}^{R} \frac{1}{1+x^4} \,dx + \int_C f(z)\,dz.$$
The contour $D$ encloses two simple poles of $f(z)$, the two in the first and second quadrant.

The residues of $f$ at these poles are

$$\text{Res}(f, z_1) = \frac{1}{4(e^{i\pi/4})^3} = \frac{1}{4e^{3i\pi/4}}$$

$$\text{Res}(f, z_2) = \frac{1}{4(e^{3i\pi/4})^3} = \frac{1}{4e^{9i\pi/4}} = \frac{1}{4e^{i\pi/4}}.$$ 

The winding numbers of $D$ at the poles are

$$l(D, z_j) = 1 \text{ for } j = 1, 2.$$
By the Residue Theorem we have

\[
\oint_D \frac{1}{1 + z^4} \, dz = 2\pi i \left[ \operatorname{Res} \left( \frac{1}{1 + z^4}, z_1 \right) + \operatorname{Res} \left( \frac{1}{1 + z^4}, z_2 \right) \right]
\]

\[
= 2\pi i \left[ \frac{1}{4e^{3i\pi/4}} + \frac{1}{4e^{i\pi/4}} \right]
\]

\[
= \frac{\pi i}{2} \left[ e^{-3i\pi/4} + e^{-i\pi/4} \right]
\]

\[
= \frac{\pi i}{2} \left[ \cos(3\pi/4) - i \sin(3\pi/4) + \cos(\pi/4) - i \sin(\pi/4) \right]
\]

\[
= \frac{\pi i}{2} \left[ -\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} \right]
\]

\[
= \frac{\pi i}{2} \left( -\frac{2i}{\sqrt{2}} \right)
\]

\[
= \frac{\pi}{\sqrt{2}}.
\]
By the parameterization $\xi(\theta) = Re^{i\theta}, \theta \in [0, \pi]$, of $C$ we obtain

\[
\left| \int_C \frac{1}{1 + z^4} \, dz \right| = \left| \int_0^\pi \frac{iRe^{i\theta}}{1 + R^4 e^{4i\theta}} \, d\theta \right|
\leq \int_0^\pi \left| \frac{iRe^{i\theta}}{1 + R^4 e^{4i\theta}} \right| \, d\theta
= \int_0^\pi \frac{R}{|1 + R^4 e^{4i\theta}|} \, d\theta
\leq \int_0^\pi \frac{R}{|R^4 e^{4i\theta}| - 1} \, d\theta
= \frac{R\pi}{R^4 - 1},
\]

where for the last inequality we have used the “reverse” triangle inequality

\[
|R^4 e^{4i\theta}| - |1| \leq |R^4 e^{4i\theta} - (1)|.
\]
Letting \( R \to \infty \) we obtain from

\[
\oint_D f(z) \, dz = \int_{-R}^{R} \frac{1}{1 + x^4} \, dx + \int_C f(z) \, dz
\]

that

\[
\int_{-\infty}^{\infty} \frac{1}{1 + x^4} \, dx = \oint_D f(z) \, dz = \frac{\pi}{\sqrt{2}}
\]

since

\[
\lim_{R \to \infty} \int_C f(z) \, dz = 0.
\]
What questions do you have?