Math 346 Lecture #12
8.1 Multivariable Integration

We extend the construction of the regulated integral for functions of a single variable to functions of several variables by defining the integral for step functions and then applying the continuous linear extension theorem.

8.1.1 Multivariable Step Functions

Definition 8.1.1. For \( a = (a_1, \ldots, a_n) \) and \( b = (b_1, \ldots, b_n) \) in \( \mathbb{R}^n \) with \( a_i \leq b_i \) for all \( i = 1, \ldots, n \), the closed \( n \)-interval \([a, b]\) is defined to be

\[
[a_1, b_1] \times \cdots \times [a_n, b_n] \subset \mathbb{R}^n.
\]

The closed \( n \)-interval \([a, b]\) is a compact box or parallelepiped in \( \mathbb{R}^n \).

Definition 8.1.2. A subdivision \( \mathcal{P} \) of a closed \( n \)-interval \([a, b]\) consists, for each \( i = 1, \ldots, n \), of a subdivision \( \mathcal{P}_i = \{t_i^{(0)} = a_i < t_i^{(1)} < \cdots < t_i^{(k_i - 1)} < t_i^{(k_i)} = b_i\} \) of the closed interval \([a_i, b_i]\) for some \( k_i \in \mathbb{N} \).

Definition 8.1.3. A subdivision of an \( n \)-interval \([a, b]\) gives a decomposition of \([a, b]\) into a disjoint union of partially open subinterval and closed subinterval as follows. Each \( t_i^{(j)} \) with \( 0 < j < k_i \) defines a hyperplane

\[
H_i^{(j)} = \{x \in \mathbb{R}^n : x_i = t_i^{(j)}\}
\]

in \( \mathbb{R}^n \) which divides \([a, b]\) into two regions; a partially open subinterval

\[
[a_1, b_1] \times \cdots \times [a_{i-1}, b_{i-1}] \times [a_i, t_i^{(j)}] \times [a_{i+1}, b_{i+1}] \times \cdots \times [a_n, b_n],
\]

and a closed subinterval

\[
[a_1, b_1] \times \cdots \times [a_{i-1}, b_{i-1}] \times [t_i^{(j)}, b_i] \times [a_{i+1}, b_{i+1}] \times \cdots \times [a_n, b_n].
\]

(Sketch the picture in \( \mathbb{R}^2 \).
)

Repeating this process for each resulting subinterval (either partially open or closed) and for each hyperplane, we get a decomposition of \([a, b]\) into a pairwise disjoint union of subintervals.

We account for each these subintervals \( R_I \) through an \( n \)-tuple \( I = (i_1, \ldots, i_n) \) where \( i_j \in \{1, \ldots, k_j\} \), which is to say that each subinterval is associated with the corner whose coordinates are

\[
(t_1^{i_1}, \ldots, t_n^{i_n}).
\]

(In \( \mathbb{R}^2 \), this point is the top right corner.)

This association gives a bijection from

\[
\{1, \ldots, k_1\} \times \cdots \times \{1, \ldots, k_n\}
\]
to $\mathcal{P}$ by which we identify $(i_1, \ldots, i_n)$ with $(t^i_1, \ldots, t^i_n)$.

All except one of the subintervals $R_I$ are partially open, i.e., have a least one face missing; the exception is the closed subinterval $R_I$ for $I = (k_1, \ldots, k_n)$.

We thus have a pairwise disjoint union of $[a, b]$ into subintervals:

$$[a, b] = \bigcup_{I \in \mathcal{P}} R_I.$$  

We assume throughout the remainder of this lecture that $(X, \| \cdot \|)$ is a Banach space over $\mathbb{R}$.

**Definition 8.1.4.** For a subset $E$ of $\mathbb{R}^n$ the indicator or characteristic function on $E$ is

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{if } x \notin E. \end{cases}$$

A function $s : [a, b] \to X$ is a step function if there exists a subdivision $\mathcal{P}$ of $[a, b]$ and elements $x_I \in X$ for each $I \in \mathcal{P}$ such that

$$s(x) = \sum_{I \in \mathcal{P}} x_I \chi_{R_I}(x).$$

More generally we consider a function $s : E \to X$ to be a step function if it is zero outside some interval $[a, b]$ and the restriction of $s$ to $[a, b]$ is a step function on $[a, b]$.

Let $S([a, b], X)$ denote the collection of all step functions $s : [a, b] \to X$. This collection is nonempty because the zero function is a step function.

**Proposition 8.1.5.** The collection $S([a, b], X)$ is a subspace of the Banach space, $(L^\infty([a, b], X), \| \cdot \|_\infty)$, of bounded functions from $[a, b]$ to $X$.

The proof of this is nearly identical to the single variable counterpart (see Proposition 5.10.3).

### 8.1.2 Multivariable, Banach-Valued Integration

Every $n$-interval has a naturally defined $n$-dimensional volume or “measure.”

**Definition 8.1.6.** For each $j \in \{1, \ldots, n\}$ let $A_j$ be an interval of one of the forms $(a_j, b_j)$, $[a_j, b_j)$, $(a_j, b_j]$, or $[a_j, b_j]$ for $a_j \leq b_j$ and $a_j, b_j \in \mathbb{R}$. We define the measure of $R = A_1 \times \cdots \times A_n$ to be

$$\lambda(R) = \prod_{j=1}^n (b_j - a_j).$$

Each $R$ with nonempty interior will have a positive measure, while any $R$ with some $a_j = b_j$ will have zero measure.

**Definition 8.1.7.** The integral of a step function

$$s(x) = \sum_{I \in \mathcal{P}} x_I \chi_{R_I}(x)$$
in $S([a, b], X)$ is
\[ \mathcal{J}(s) = \int_{[a, b]} s = \sum_{I \in \mathcal{I}} x_I \lambda(R_I), \]
a finite linear combination in the Banach space $X$.

**Proposition 8.1.8.** For any $n$-interval $[a, b] \subset \mathbb{R}^n$ (which $n$-interval is compact by definition), the integral operator $\mathcal{J} : S([a, b], X) \to X$ is a bounded linear transformation where
\[ \|\mathcal{J}\| = \lambda([a, b]). \]
The proof of this is HW (Exercise 8.2). [Notice that there is a typo in the book: $\lambda([b-a])$ should be $\lambda([a, b])$. This typo appears again in Theorem 8.1.9.]

**Note.** Recall that we showed in the lecture note for Section 5.7 that the closure of a subspace is a subspace (a result not mentioned nor proved in the book).

**Theorem 8.1.9 (Multivariable, Banach-Valued Integral).** The bounded linear transformation $\mathcal{J} : S([a, b], X) \to X$ extends uniquely to a bounded linear transformation $\mathcal{J} : S([a, b], X) \to X$ such that
\[ \|\mathcal{J}\| = \lambda([a, b]). \]
Moreover we have
\[ C([a, b], X) \subset S([a, b], X) \subset L^\infty([a, b], X). \]
The proof of this is HW (Exercise 8.3).

**Definition 8.1.10.** For any compact $n$-interval $[a, b]$ we denote the set $S([a, b], X)$ by $\mathcal{B}([a, b], X)$ which means the closed subspace of regulated-integrable functions.

For $f \in \mathcal{B}([a, b], X)$ we call the the bounded linear transformation $\mathcal{J}$ the integral of $f$ and write
\[ \int_{[a, b]} f = \mathcal{J}(f). \]

**Proposition 8.1.11.** For $f, g \in \mathcal{B}([a, b], X)$, the following hold.

(i) \[ \left\| \int_{[a, b]} f \right\| \leq \lambda([a, b]) \sup_{t \in [a, b]} \|f(t)\|. \]

(ii) For a sequence $(f_n)_{n=1}^\infty$ in $\mathcal{B}([a, b], X)$ converging uniformly to $f$ there holds
\[ \lim_{n \to \infty} \int_{[a, b]} f_n = \int_{[a, b]} \lim_{n \to \infty} f_n = \int_{[a, b]} f. \]

(iii) With $\|f\|$ denoting the function $t \to \|f(t)\|$ from $[a, b]$ to $\mathbb{R}$, there holds
\[ \left\| \int_{[a, b]} f \right\| \leq \int_{[a, b]} \|f\|. \]
(iv) If \( \|f(t)\| \leq \|g(t)\| \) for all \( t \in [a, b] \), then
\[
\int_{[a,b]} \|f\| \leq \int_{[a,b]} \|g\|. 
\]

Proof. Parts (i) and (ii) are HW (Exercise 8.5).

(iii) For a step function
\[
s(t) = \sum_{I \in \mathcal{P}} x_I \chi_{R_I}(t) \in \mathcal{R}([a, b], X)
\]
we have by the pairwise disjointness of the \( R_I \) that
\[
\|s(t)\| = \sum_{I \in \mathcal{P}} \|x_I\| \chi_{R_I}(t) \in \mathcal{R}([a, b], \mathbb{R}).
\]

Since \( s \) is a finite sum, we have by the triangle inequality that
\[
\left\| \int_{[a,b]} s \right\| = \left\| \sum_{I \in \mathcal{P}} x_I \lambda(R_I) \right\| \leq \sum_{I \in \mathcal{P}} \|x_I\| \lambda(R_I) = \int_{[a,b]} \|s\|.
\]

For any \( f \in \mathcal{R}([a, b], X) \) there is a sequence of step functions \( (s_n)_{n=1}^\infty \) such that \( s_n \to f \) uniformly on \( [a, b] \).

This implies by the continuity of the norm and part (ii) that
\[
\left\| \int_{[a,b]} f \right\| = \left\| \int_{[a,b]} \lim_{n \to \infty} s_n \right\|
\leq \lim_{n \to \infty} \left\| \int_{[a,b]} s_n \right\|
= \lim_{n \to \infty} \int_{[a,b]} \|s_n\|
= \int_{[a,b]} \left\| \lim_{n \to \infty} s_n \right\|
= \int_{[a,b]} \|f\|.
\]

(iv) Suppose \( h \in \mathcal{R}([a, b], \mathbb{R}) \) satisfies \( h(t) \geq 0 \) for all \( t \in [a, b] \).

There is a sequence of step functions \( (s_n)_{n=1}^\infty \) that converges uniformly to \( f \) on \( [a, b] \): for \( \epsilon > 0 \) there is \( N \in \mathbb{N} \) such that for all \( n \geq N \) there holds
\[
\|s_n - h\|_\infty \leq \frac{\epsilon}{\lambda([a,b])}.
\]

Since \( h(t) \geq 0 \) for all \( t \in [a, b] \), the uniform convergence implies for all \( n \geq N \) that
\[
\frac{\epsilon}{\lambda([a,b])} \geq |h(t) - s_n(t)| \geq h(t) - s_n(t) \geq -s_n(t) \text{ for all } t \in [a, b].
\]
This implies for all $n \geq N$ that
\[ s_n(t) \geq -\frac{\epsilon}{\lambda([a, b])} \text{ for all } t \in [a, b]. \]

Consequently, as
\[ s_n(t) = \sum_{I \in \mathcal{P}} x_I \chi_{R_I}(t) \]
for $x_I \in \mathbb{R}$, it follows for all $I \in \mathcal{P}$ that
\[ x_I \geq -\frac{\epsilon}{\lambda([a, b])}. \]

Hence for each $n \geq N$ we have
\[
\int_{[a, b]} s_n = \sum_{I \in \mathcal{P}} x_I \lambda(R_I) \geq -\sum_{I \in \mathcal{P}} \frac{\epsilon \lambda(R_I)}{\lambda([a, b])} \\
= -\frac{\epsilon}{\lambda([a, b])} \sum_{I \in \mathcal{P}} \lambda(R_I) \\
= -\frac{\epsilon}{\lambda([a, b])} \lambda([a, b]) \\
= -\epsilon.
\]

By part (ii) we have for all $n \geq N$ that
\[
\int_{[a, b]} h = \lim_{n \to \infty} \int_{[a, b]} s_n \geq -\epsilon.
\]

Since this holds for any $\epsilon > 0$ we conclude that
\[
\int_{[a, b]} h \geq 0.
\]

By setting $h(t) = \|f(t)\| - \|g(t)\|$ we obtain the result. \(\square\)

**Remark 8.1.12.** The Riemann construction of the integral defines a bounded linear transformation on $\mathcal{R}([a, b], X)$ that agrees with the regulated integral on step functions. Hence by the uniqueness part of the Continuous Linear Extension Theorem, the Riemann integral and the regulated integral agree on $\mathcal{R}([a, b], X)$.

**8.1.3 Integration over subsets of $[a, b]$**

To integrate functions defined on bounded subsets $E$ of $\mathbb{R}^n$ other than closed $n$-intervals, we extend the functions by zero outside of $E$.

**Definition 8.1.13.** For any function $f : E \to X$, the extension of $f$ by zero is the function
\[
f \chi_E(z) = \begin{cases} f(z) & \text{if } z \in E, \\ 0 & \text{if } z \notin E. \end{cases}
\]
Since $E$ is bounded in $\mathbb{R}^n$, its closure is compact, and there is a compact $n$-interval $[a, b]$ that contains $E$.

We could then define the integral of $f$ to be

$$\int_E f = \int_{[a,b]} f \chi_E.$$ 

An immediate problem with doing this is that we don’t know beforehand if $f \chi_E$ belongs to $\mathcal{R}([a, b], X)$.

It is even possible that the indicator function $\chi_E$ may not be integrable.

**Unexample 8.1.14.** For an compact 1-interval $[a, b]$ with $a < b$, the singleton set $E = \{p\}$ for $p \in [a, b]$ has $\chi_E$ not integrable.

This follows because every step function $s : [a, b] \to \mathbb{R}$ is right continuous, meaning for every $t_0 \in [a, b]$ there holds

$$\lim_{t \to t_0^+} s(t) = s(t_0).$$

By Exercise 8.4 (a HW problem) the uniform limit of right-continuous functions is a right-continuous function.

But the indicator function $\chi_E$ is not right-continuous at $t_0 = p$, and therefore is not integrable.

Overcoming this and other deficiencies of the regulated integral is discussed in the next section.