11.6 Power Series and Laurent Series

We have already seen that a convergent power series \( f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n \) gives a holomorphic function on \( B(z_0, r) \) for some \( r > 0 \). In this lecture we see the converse of this: every function \( f \) holomorphic on an open set \( U \) can be written as a convergent power series on \( B(z_0, r) \subset U \) for every \( z_0 \in U \) and some \( r > 0 \). We will also see that there is an extension of power series, called Laurent series, that gives a way to write a function that is holomorphic on a punctured ball or disk \( B(z_0, r) \setminus \{z_0\} \), that describes the kind of singularity the function has at \( z_0 \).

Throughout we assume that \((X, \| \cdot \|_X)\) is a complex Banach space.

11.6.1 Power Series

For a function \( f : U \to X \) holomorphic on an open \( U \) in \( \mathbb{C} \), we can form for each \( z_0 \in U \) the Taylor series

\[
\sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k
\]

because \( f \) is infinitely holomorphic on \( U \) by Corollary 11.4.8. Using Cauchy’s Integral and Differentiation formulas we show that the Taylor series of \( f \) at \( z_0 \) converges to \( f \) on some open ball \( B(z_0, r) \subset U \) for \( r > 0 \).

**Theorem 11.6.1.** For an open \( U \) in \( \mathbb{C} \), if \( f : U \to X \) is holomorphic, then for each \( z_0 \in U \) there exists a largest \( r \in (0, \infty] \) such that \( B(z_0, r) \subset U \) and the Taylor series for \( f \) at \( z_0 \) converges uniformly to \( f \) on compact subset of \( B(z_0, r) \).

**Proof.** For a fixed \( z_0 \in U \), there is a largest \( r > 0 \) by the openness of \( U \) such that \( B(z_0, r) \subset U \).

For any \( 0 < \epsilon < r \) consider the compact set \( D = B(z_0, r - \epsilon) \subset B(z_0, r) \).

To get uniform convergence of the Taylor series on compact subset of \( B(z_0, r) \) is suffices to show uniform convergence on sets of the form \( D \).

The circle \( \gamma \) given by

\[
\{w \in \mathbb{C} : |w - z_0| = r - \epsilon/2\}
\]

lies in \( B(z_0, r) \setminus D \), so that \( \gamma \) encloses \( D \).

We orient \( \gamma \) with the positive orientation.

For any \( z \in D \) and any \( w \in \gamma \) we have \( |z - z_0| \leq r - \epsilon \) and \( |w - z_0| = r - \epsilon/2 \); thus

\[
\left| \frac{z - z_0}{w - z_0} \right| = \frac{|z - z_0|}{|w - z_0|} \leq \frac{r - \epsilon}{r - \epsilon/2} < 1.
\]

This implies convergence of the geometric series in \((z - z_0)/(w - z_0)\),

\[
\frac{1}{1 - \frac{z - z_0}{w - z_0}} = \sum_{k=0}^{\infty} \left( \frac{z - z_0}{w - z_0} \right)^k = \sum_{k=0}^{\infty} \frac{1}{(w - z_0)^k} (z - z_0)^k.
\]
This convergence is uniform and absolute on $D$ for each $w \in \gamma$ by the Lemma 11.2.5 (Abel-Weierstrass Lemma) because for $a_k = 1/(w - z_0)^k$, $R = |w - z_0|$, and $M = 1$ there holds

$$|a_k| R^k = \frac{1}{|w - z_0|^k} |w - z_0|^k = 1 \leq M.$$  

[We get $|a_k| = 1/|w - z_0|^k$ by applying the norm to $(w - z_0)^k a_k = 1.$]

Since

$$(w - z_0) \left(1 - \frac{z - z_0}{w - z_0}\right) = w - z_0 - (z - z_0) = w - z$$

we obtain

$$\frac{1}{w - z} = \frac{1}{w - z_0} \frac{1}{1 - \frac{z - z_0}{w - z_0}} = \frac{1}{w - z_0} \sum_{k=0}^{\infty} \left(\frac{z - z_0}{w - z_0}\right)^k = \sum_{k=0}^{\infty} \left(\frac{z - z_0}{w - z_0}\right)^{k+1}.$$  

Since $\gamma$ is compact, the restriction of the continuous $f$ to $\gamma$ is bounded; this implies that

$$\frac{f(w)}{w - z} = \sum_{k=0}^{\infty} \frac{f(w)(z - z_0)^k}{(w - z_0)^{k+1}}$$

is also uniformly and absolutely convergent on $D$ for each $w \in \gamma$.

By the Cauchy Integral formula we have that

$$f(z) = \frac{1}{2\pi i} \oint_{\gamma} f(w) \frac{1}{w - z} dw = \frac{1}{2\pi i} \oint_{\gamma} \left(\sum_{k=0}^{\infty} \frac{f(w)(z - z_0)^k}{(w - z_0)^{k+1}}\right) dw.$$  

Since integration is a continuous linear transformation, it commutes with uniform limits, which means in particular that integration commutes with uniformly convergent sums:

$$f(z) = \sum_{k=0}^{\infty} \frac{1}{2\pi i} \oint_{\gamma} \frac{f(w)(z - z_0)^k}{(w - z_0)^{k+1}} dw = \sum_{k=0}^{\infty} \left(\frac{1}{2\pi i} \oint_{\gamma} \frac{f(w)}{(w - z_0)^{k+1}} dw\right) (z - z_0)^k.$$  

By Cauchy’s Differentiation formula we have

$$\frac{f^{(k)}(z_0)}{k!} = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(w)}{(w - z_0)^{k+1}} dw.$$  

This gives

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k$$

for all $z \in D$, i.e., we have uniformly and absolute convergence of the Taylor series for $f$ at $z_0$ to $f$ on compact subsets of $B(z_0, r)$.  

\[\Box\]

Remark 11.6.2. We now have shown that a function $f : U \to X$ is holomorphic on $U$ if and only if $f$ is analytic on $U$. Because of this equivalence, we often use holomorphic and analytic interchangeably.
Proposition 11.6.3. For \( \{a_k\}_{k=0}^{\infty} \subset X \), a convergent power series

\[
f(z) = \sum_{k=0}^{\infty} a_k(z - z_0)^k
\]

about \( z_0 \in \mathbb{C} \) is unique and equal to its Taylor series.

Proof. By Theorem 11.2.8, the convergent power series is differentiable with derivative given by the term-by-term derivative of the power series, i.e.,

\[
f'(z) = \sum_{k=1}^{\infty} ka_k(z - z_0)^k
\]

with the same radius of convergence \( R \) as the original power series.

By induction this shows that the \( j^{\text{th}} \) derivative of \( f \) is given by the power series

\[
f^{(j)}(z) = \sum_{k=j}^{\infty} k(k-1) \cdots (k-j+1)a_k(z - z_0)^{k-j}
\]

with the same radius of convergence \( R \) as the original power series. Thus we obtain for every \( j = 0, 1, 2, \ldots \), that

\[
f^{(j)}(z_0) = j!a_j \text{ or } a_j = \frac{f^{(j)}(z_0)}{j!}.
\]

This implies that \( f \) is equal to its Taylor series at \( z_0 \), i.e.,

\[
f(z) = \sum_{j=0}^{\infty} \frac{f^{(j)}(z_0)}{j!}(z - z_0)^j
\]

for all \( z \in B(z_0, R) \).

Since there is only one Taylor series of \( f \) about \( z_0 \), we obtain the uniqueness of the power series of \( f \) about \( z_0 \).

Corollary 11.6.4. For an open, path-connected subset \( U \) of \( \mathbb{C} \), and a holomorphic \( f : U \to X \), if there exists \( z_0 \in U \) such that \( f^{(n)}(z_0) = 0 \) for all \( n = 0, 1, 2, 3, \ldots \), then \( f(z) = 0 \) for all \( z \in U \).

Proof. By way of contradiction, assume the hypothesis and the existence of \( w \in U \) such that \( f(w) \neq 0 \).

The Taylor series of \( f \) about \( z_0 \) is the zero function because

\[
f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k = \sum_{k=0}^{\infty} 0(z - z_0)^k = 0
\]

for all \( z \in B(z_0, R) \) where \( R \) is the largest \( R > 0 \) such that \( B(z_0, R) \subset U \).
By the path-connectedness of $U$ there is a path $\gamma : [a, b] \to U$ such that $\gamma(a) = z_0$ and $\gamma(b) = w$.

Following the same argument found in the proof of the Precursor of the Maximum Modulus Principle, for $\epsilon = d(\gamma, U^c) > 0$ there is a finite collection of points $z_0, z_1, \ldots, z_n = w$ ordered along the path $\gamma$ such that $d(z_{j+1}, z_j) < \epsilon/2$ for all $j = 1, 2, \ldots, n$, each ball $B(z_j, \epsilon) \subset U$, and $z_j \in B(z_{j+1}, \epsilon)$ for all $j = 0, 1, \ldots, n - 1$.

By Theorem 11.6.4, the holomorphic function has a convergent power series about each $B$, ordered along the path $\gamma$.

By Proposition 11.6.3, there holds $f(z) = 0$ for all $z \in B(z_0, \epsilon)$, then $f^{(k)}(z_1) = 0$ for all $k = 0, 1, 2, \ldots$.

By Proposition 11.6.3, there holds $f(z) = 0$ for all $z \in B(z_1, \epsilon)$.

Continuing this argument we arrive at $f(z) = 0$ for all $z \in B(w, \epsilon)$, which contradicts $f(w) \neq 0$.

\section*{11.6.2 Zeros of Analytic Functions}

Another property of a not identically equal to zero holomorphic function $f : U \to X$ is that its zeros, if any, must be isolated, i.e., if $f(z_0) = 0$, then there exists $\epsilon > 0$ such that $f(z) \neq 0$ for all $z \in B(z_0, \epsilon) \setminus \{z_0\}$. We will use the order of a zero (defined next) to obtain this isolation of zeros.

\textbf{Definition 11.6.5.} For an open $U$, we say that a holomorphic $f : U \to X$ has a zero of order $n \in \mathbb{N}$ at $z_0 \in U$ if the Taylor series of $f$ about $z_0$ has the form

$$f(z) = \sum_{k=n}^{\infty} a_k(z - z_0)^k$$

for $a_n \neq 0$, i.e., $f^{(j)}(z_0) = 0$ for all $j = 0, 1, \ldots, n - 1$ and $f^{(n)}(z_0) \neq 0$.

\textbf{Proposition 11.6.6.} For an open $U$ and $f : U \to X$ holomorphic, if $z_0 \in U$ is a zero of order $n$ for $f$, then there exists a holomorphic function $g : U \to X$ such that $f(z) = (z - z_0)^n g(z)$ and $g(z_0) \neq 0$, and there exists $\epsilon > 0$ such that $B(z_0, \epsilon) \subset U$ and $f(z) \neq 0$ for all $z \in B(z_0, \epsilon) \setminus \{z_0\}$.

\textbf{Proof.} From the convergent power series we have

$$f(z) = \sum_{k=n}^{\infty} a_k(z - z_0)^k = (z - z_0)^n \sum_{k=n}^{\infty} a_k(z - z_0)^{k-n} = (z - z_0)^n \sum_{j=0}^{\infty} a_{k+n}(z - z_0)^j,$$

where the last equality is a consequence of the change of index $j = k - n$.

The function

$$g(z) = \sum_{j=0}^{\infty} a_{k+n}(z - z_0)^j$$

is a convergent power series and hence holomorphic.
By hypothesis, \( a_n \neq 0 \), so that \( g(z_0) = a_n \neq 0 \).

By the implied continuity of \( g \) at \( z_0 \), there exists \( \epsilon > 0 \) such that \( B(z_0, \epsilon) \subset U \) and \( g(z) \neq 0 \) for all \( z \in B(z_0, \epsilon) \).

Since the polynomial \((z - z_0)^n\) only equals zero when \( z = z_0 \), the function \( f(z) = (z - z_0)^n g(z) \) does not vanish (meaning does not equal zero) for all \( z \in B(z_0, \epsilon) \setminus \{z_0\} \).

**Corollary 11.6.7 (Local Isolation of Zeros).** For an open, path-connected \( U \) in \( \mathbb{C} \), and a holomorphic \( f : U \to X \), if there is a sequence \((z_k)_{k=1}^{\infty}\) of distinct points in \( U \) where \( z_k \to w \in U \) and \( f(z_k) = 0 \) for all \( k \in \mathbb{N} \), then \( f(z) = 0 \) for all \( z \in U \).

**Proof.** By way of contradiction, suppose that there is a sequence \((z_k)_{k=1}^{\infty}\) of distinct points in \( U \) where \( z_k \to w \in U \) and \( f(z_k) = 0 \) for all \( k \), while there exists \( z \in U \) such that \( f(z) \neq 0 \).

Suppose there exists some \( \nu > 0 \) with \( B(w, \nu) \subset U \) such that \( f(z) = 0 \) for all \( z \in B(w, \nu) \).

Then \( f^{(j)}(w) = 0 \) for all \( j = 0, 1, 2, \ldots \), hence by the path-connectedness of \( U \) and Corollary 11.6.4 we obtain \( f(z) = 0 \) for all \( z \in U \).

This contradicts the existence of \( z \in U \) for which \( f(z) \neq 0 \).

Hence for all \( \nu > 0 \) with \( B(w, \nu) \subset U \) there exists \( z \in B(w, \nu) \) such that \( f(z) \neq 0 \).

Fix \( \nu > 0 \) to be the largest value for which \( B(w, \nu) \subset U \).

By the convergence of \( z_k \to w \) and the continuity of \( f \) at \( w \), we have that \( f(w) = 0 \).

Since there is \( z \in B(w, \nu) \) for which \( f(z) \neq 0 \), the Taylor series for \( f \) about \( w \) is not identically zero, meaning there exists a smallest \( n \in \mathbb{N} \) such that \( f^{(n)}(w) \neq 0 \).

Thus \( w \) is a zero of order \( n \) for \( f \).

By Proposition 11.6.6 applied to the zero \( w \) of order \( n \), there exists \( \epsilon \in (0, \nu) \) for which \( f(z) \neq 0 \) for all \( z \in B(w, \epsilon) \setminus \{w\} \).

By the convergence \( z_k \to w \) there exists \( N \in \mathbb{N} \) such that for all \( k \geq N \) there holds \( z_k \in B(w, \epsilon) \).

But since \( z_k \in B(w, \epsilon) \) for all \( k \geq N \) where \( f(z_k) = 0 \), this contradicts \( f(z) \neq 0 \) for all \( z \in B(w, \epsilon) \setminus \{w\} \).

**11.6.3 Laurent Series**

For a holomorphic \( f : U \to X \) such that \( f(z_0) \neq 0 \) for \( z_0 \in U \), the function

\[
g(z) = \frac{f(z)}{z - z_0}
\]

is not complex differentiable at \( z_0 \) and so there is no Taylor series for \( g \) about \( z_0 \).

We know by Cauchy’s Integral formula that for any simple closed contour \( \gamma \) in \( U \) enclosing \( z_0 \) there holds

\[
f(z_0) = \frac{1}{2\pi i} \oint_\gamma g(z) \, dz = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z - z_0} \, dz.
\]
Using the Taylor’s series for $f$ about $z_0$, i.e.,

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k,$$

we can express the function $g$ as

$$g(z) = \frac{f(z)}{z-z_0} = \frac{1}{z-z_0} \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^{k-1}$$

which makes sense on the open $U \setminus \{z_0\}$.

Since $z_0 \not\in \gamma$ and integration and uniform convergence commute, we can use the series expression for $g$ to compute

$$\oint_{\gamma} g(z) \, dz = \oint_{\gamma} \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^{k-1} = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} \oint_{\gamma} (z-z_0)^{k-1}.$$

By Lemma 11.3.5, the contour integral $\oint_{\gamma} (z-z_0)^{k-1} \, dz = 0$ when $k \geq 1$, and $\oint_{\gamma} (z-z_0)^{k-1} \, dz = 2\pi i$ when $k = 0$.

Thus we obtain

$$\frac{1}{2\pi i} \oint_{\gamma} g(z) \, dz = \frac{2\pi i}{2\pi i} f(z_0) = f(z_0)$$

in agreement with Cauchy’s Integral Formula.

**Definition.** For coefficients $a_k \in X$ for $k \in \mathbb{Z}$, a Laurent series is a series of the form

$$\sum_{k=-\infty}^{\infty} a_k(z-z_0)^k.$$

To talk about convergence of Laurent series we will use the open annulus $A$ centered at $z_0$ with inner radius $r$ and outer radius $R$ defined by

$$A = \{z \in \mathbb{C} : r < |z-z_0| < R\}$$

where $0 \leq r < R \leq \infty$.

**Theorem 11.6.8 (Laurent Expansion).** For the open annulus $A$ centered at $z_0$ with inner radius $r$ and outer radius $R$, if $f : A \to X$ is holomorphic, then $f$ has a Laurent series

$$f(z) = \sum_{k=-\infty}^{\infty} a_k(z-z_0)^k$$

where each of the power series in the decomposition

$$\sum_{k=0}^{\infty} a_k(z-z_0)^k + \sum_{k=1}^{\infty} a_{-k} \left(\frac{1}{z-z_0}\right)^k$$
converge uniformly and absolutely on every compact subannulus
\[ D_{\rho, \varrho} = \{ z \in \mathbb{C} : \rho \leq |z - z_0| \leq \varrho \} \]
of \( A \), i.e., for all \( r < \rho < \varrho < R \). The coefficients \( a_k \) in the Laurent series for \( f \) are given explicitly by
\[ a_k = \frac{1}{2\pi i} \oint_{\gamma \neq 0} \frac{f(w)}{(w - z_0)^{k+1}} \, dw \]
for any circle \( \gamma \) of radius strictly between \( r \) and \( R \).

See the book for the proof.

**Proposition 11.6.9.** For an open annulus centered at \( z_0 \) with inner radius \( r \) and outer radius \( R \), the Laurent series for a holomorphic \( f : A \to X \) is unique.

The proof of this is HW (Exercise 11.28). The start of the proof is given in the book.

**Remark 11.6.10.** Computing the Laurent series is usually quite difficult. As we will see, the only coefficient we really need in the Laurent series of \( f \) holomorphic on the annulus \( A = B(z_0, \epsilon) \setminus \{0\} \) is that of the term \((z - z_0)^{-1}\) when computing any contour integral \( \oint f(z) \, dz \) for a simple closed contour in \( A \) that encloses \( z_0 \).

**Example (in lieu of 11.6.11).** The Laurent series of \( f(z) = \sin(z)/z^4 \) on the open annulus \( A = \{ z \in \mathbb{C} : 0 < |z| < \infty \} = \mathbb{C} \setminus \{0\} \) is obtained by dividing the power series for \( \sin(z) \) by \( z^4 \), i.e.,
\[ \frac{1}{z^4} \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k-3}}{(2k+1)!} = \frac{1}{3} z - \frac{1}{6} z^3 + \frac{z}{5!} - \cdots. \]

Since the Laurent series for this holomorphic function converges uniformly on \( A \), we can compute the contour integral of \( f \) over any circle centered at \( 0 \) with radius \( \nu > 0 \) by a “direct” calculation after commuting the sum and the integral:
\[ \oint_{\gamma} f(z) \, dz = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \oint_{\gamma} (z - z_0)^{2k-3} \, dz = -\frac{1}{6} \oint_{\gamma} (z - z_0)^{-1} \, dz = -\frac{\pi i}{3}, \]
where all the other contour integrals are zero by Lemma 11.3.5.

This gives
\[ \frac{1}{2\pi i} \oint_{\gamma} f(z) \, dz = -\frac{1}{6}. \]

On the other hand, by Cauchy’s Differentiation formula we arrive at the same answer:
\[ \frac{1}{2\pi i} \oint_{\gamma} f(z) \, dz = \frac{1}{3!} \frac{3!}{2\pi i} \oint_{\gamma} \frac{\sin(z)}{(z - 0)^4} \, dz = \frac{1}{6} \left( -\cos(0) \right) = -\frac{1}{6} \]
because the third derivative of \( \sin(z) \) is \( -\cos(z) \).

**Example (in lieu of 11.6.12).** Find the Laurent series for
\[ f(z) = \frac{2}{(z - 1)^2(z + 1)} \]
about the point $z_0 = 1$, i.e., an open annulus centered at $z_0 = 1$. [We will determine the inner and outer radius in a moment.]

Applying the method of partial fractions to the function gives

$$f(z) = \frac{1}{(z-1)^2} - \frac{1/2}{z-1} + \frac{1/2}{z+1}.$$  

We express the last term as a power series in $(z-1)$ using the geometric series as follows:

$$\frac{1/2}{z+1} = \frac{1/2}{2 - (-z+1)} = \frac{1/4}{1 - (-z+1)/2} = \frac{1}{4} \sum_{k=0}^{\infty} \left(\frac{-z+1}{2}\right)^k = \frac{1}{4} \sum_{k=0}^{\infty} \left(-\frac{z-1}{2}\right)^k = \frac{1}{4} \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k} (z-1)^k = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{k+2}} (z-1)^k.$$

We obtain the Laurent series

$$f(z) = \frac{1}{(z-1)^2} - \frac{1/2}{z-1} + \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{k+2}} (z-1)^k$$

on the open annulus $A$ centered at $z_0 = 1$ with inner radius $r = 0$ and outer radius $R = 2$ as determined by the condition for convergence of the geometric series $|(-z+1)/2| < 1$.

For a simple closed contour $\gamma$ in $A$ that encloses $z_0 = 1$ we use the Laurent series to compute

$$\frac{1}{2\pi i} \oint_{\gamma} f(z) \, dz = -\frac{1}{4\pi i} \oint_{\gamma} \frac{1}{z-1} \, dz = -\frac{2\pi i}{4\pi i} = -\frac{1}{2},$$

where we have used the interchange of integration and uniform convergence, and Lemma 11.3.5.