The goal of this and the next section is to establish the existence of a spectral decomposition of any linear operator on $\mathbb{C}^n$. Such a spectral decomposition depends on the spectral projections which are the residues of the resolvent at the eigenvalues. But such a spectral decomposition also depends, as we saw in the last lecture, on other terms in the Laurent series of the resolvent about the eigenvalues. We begin to develop a better understanding of the Laurent series of the resolvent in this section.

For $A \in M_n(\mathbb{C})$ and $\lambda \in \sigma(A)$, there exist $A_k \in M_n(\mathbb{C})$, $k \in \mathbb{Z}$, (depending on $\lambda$) such that the resolvent of $A$ as a Laurent series about $\lambda$ has the form

$$R_A(z) = \sum_{k=-\infty}^{\infty} A_k (z - \lambda)^k.$$

By the Laurent Expansion Theorem, we have for each $k \in \mathbb{Z}$ that

$$A_k = \frac{1}{2\pi i} \oint_{\Gamma} \frac{R_A(z)}{(z - \lambda)^{k+1}} \, dz$$

for a positively oriented simple closed contour $\Gamma$ enclosing $\lambda$ but no other element of $\sigma(A)$. The coefficient $A_{-1}$ is the spectral projection $P_\lambda$. We are going to discover the nature of the relationships that exist among all the coefficient matrices $A_k$ in Laurent series for $R_A(z)$ about $\lambda$.

Nota Bene 12.5.1. Be aware that $A_k$ is a coefficient matrix in a Laurent series while $A^k$ is the $k^{th}$ power of $A$.

Notation. For $n \in \mathbb{Z}$, define

$$\eta_n = \begin{cases} 
1 & \text{if } n \geq 0, \\
0 & \text{if } n < 0. 
\end{cases}$$

This is a characteristic or indicator function on the set $\mathbb{Z}$.

Lemma 12.5.2. For $A \in M_n(\mathbb{C})$ and $\lambda \in \sigma(A)$, let $\Gamma$ and $\Gamma'$ be two positively oriented simple closed contours in $\rho(A)$ enclosing $\lambda$ and no other element of $\sigma(A)$. Assume further that $\Gamma$ is in the interior of $\Gamma'$, that $z' \in \Gamma'$, and $z \in \Gamma$. Then for all $m \in \mathbb{N}$ there holds

(i) $$\frac{1}{2\pi i} \oint_{\Gamma} (z - \lambda)^{-m-1}(z' - z)^{-1} \, dz = \eta_m(z' - \lambda)^{-m-1},$$

and for all $n \in \mathbb{N}$ there holds

(ii) $$\frac{1}{2\pi i} \oint_{\Gamma'} (z' - \lambda)^{-n-1}(z' - z)^{-1} \, dz' = (1 - \eta_n)(z - \lambda)^{-n-1}.$$

Proof. (i) Since every point $z'$ on $\Gamma'$ is outside of $\Gamma$, the function $(z' - z)^{-1}$ is holomorphic within $\Gamma$. 

Using the geometric series, we expand \((z' - z)^{-1}\) in terms of \(z - \lambda\):

\[
\frac{1}{z' - z} = \frac{1}{z' - \lambda} \cdot \frac{1}{1 - \left( \frac{z - \lambda}{z' - \lambda} \right)} = \frac{1}{z' - \lambda} \sum_{k=0}^{\infty} \left( \frac{z - \lambda}{z' - \lambda} \right)^k = \sum_{k=0}^{\infty} \left( \frac{z - \lambda}{z' - \lambda} \right)^{k+1}.
\]

WLOG we may shrink \(\Gamma\) to a small circle \(\Gamma_{\lambda}\) centered at \(\lambda\) with every point on \(\Gamma_{\lambda}\) closer to \(\lambda\) than \(z'\).

For a fixed \(m \in \mathbb{N}\) we then have

\[
\frac{1}{2\pi i} \oint_{\Gamma_{\lambda}} (z - \lambda)^{-m-1}(z' - z)^{-1} \, dz = \frac{1}{2\pi i} \oint_{\Gamma_{\lambda}} (z - \lambda)^{-m-1} \left[ \sum_{k=0}^{\infty} \left( \frac{z - \lambda}{z' - \lambda} \right)^{k+1} \right] \, dz
\]

\[
= \sum_{k=0}^{\infty} (z' - \lambda)^{-k-1} \left[ \frac{1}{2\pi i} \oint_{\Gamma_{\lambda}} (z - \lambda)^{-m-1+k} \, dz \right].
\]

By Lemma 11.3.5 we have

\[
\frac{1}{2\pi i} \oint_{\Gamma_{\lambda}} (z - \lambda)^{-m-1+k} \, dz = \begin{cases} 
1 & \text{if } k = m, \\
0 & \text{if } k \neq m.
\end{cases}
\]

When \(m < 0\), the case \(k = m\) never occurs since \(k \geq 0\), thus giving

\[
\frac{1}{2\pi i} \oint_{\Gamma_{\lambda}} (z - \lambda)^{-m-1}(z' - z)^{-1} \, dz = 0.
\]

When \(m \geq 0\), only one of the integrals is nonzero, and that occurs when \(k = m\), thus giving

\[
\frac{1}{2\pi i} \oint_{\Gamma_{\lambda}} (z - \lambda)^{-m-1}(z' - z)^{-1} \, dz = (z' - \lambda)^{-m-1}.
\]

These two outcomes for the integral combine through the function \(\eta_m\) to give

\[
\frac{1}{2\pi i} \oint_{\Gamma_{\lambda}} (z - \lambda)^{-m-1}(z' - z)^{-1} \, dz = \eta_m(z' - \lambda)^{-m-1}.
\]

(ii) In this case, both \(\lambda\) and \(z\) lies inside \(\Gamma'\).

Using the Cauchy-Goursat Theorem, the same circle \(\Gamma_{\lambda}\) from part (i), a small circle \(\Gamma_z\) centered at \(z\) such that \(\Gamma_z\) does not intersect \(\Gamma'\), and the appropriate cuts, we have

\[
\frac{1}{2\pi i} \oint_{\Gamma_{\lambda}} (z' - \lambda)^{-n-1}(z' - z)^{-1} \, dz' = \frac{1}{2\pi i} \oint_{\Gamma_{\lambda}} (z' - \lambda)^{-n-1}(z' - z)^{-1} \, dz'
\]

\[
+ \frac{1}{2\pi i} \oint_{\Gamma_z} (z' - \lambda)^{-n-1}(z' - z)^{-1} \, dz'.
\]
For the former integral we have

\[
\frac{1}{2\pi i} \oint_{\Gamma_{\lambda}} (z' - \lambda)^{-n-1}(z' - z)^{-1} dz' = -\frac{1}{2\pi i} \oint_{\Gamma_{\lambda}} (z' - \lambda)^{-n-1}(z - z')^{-1} dz' = -\eta_n(z - \lambda)^{-n-1}
\]

by an argument similar to that in part (i) with \(\Gamma\) replaced with \(\Gamma'\) and \(z\) and \(z'\) switched.

For the latter integral we have by the Cauchy Integral formula that

\[
\frac{1}{2\pi i} \oint_{\Gamma_z} \frac{(z' - \lambda)^{-n-1}}{z' - z} d\lambda' = (z - \lambda)^{-n-1}
\]

since \(z' \to (z' - \lambda)^{-n-1}\) is holomorphic on a simply connected open set containing \(\Gamma_z\).

Combining the two integrals gives the result. \(\square\)

**Lemma 12.5.3.** The matrix coefficients \(A_k\) in the Laurent expansion

\[
R_A(z) = \sum_{k=-\infty}^{\infty} A_k(z - \lambda)^k
\]

about \(\lambda \in \sigma(A)\) satisfy

\[A_m A_n = (1 - \eta_m - \eta_n) A_{m+n+1}.
\]

**Proof.** Let \(\Gamma\) and \(\Gamma'\) be two positively oriented simply closed contours enclosing \(\lambda \in \sigma(A)\) but no other element of \(\sigma(A)\), and further assume that \(\Gamma'\) is in the interior of \(\Gamma'\).

For \(m, n \in \mathbb{N}\) we write

\[
A_m = \frac{1}{2\pi i} \oint_{\Gamma} \frac{R_A(z)}{(z - \lambda)^{m+1}} dz, \quad A_n = \frac{1}{2\pi i} \oint_{\Gamma'} \frac{R_A(z')}{(z' - \lambda)^{n+1}} d\lambda.'
\]

Using properties of the resolvent in Lemma 12.3.5, and Fubini’s Theorem, we have

\[
A_m A_n = \left( \frac{1}{2\pi i} \right)^2 \oint_{\Gamma} \oint_{\Gamma'} (z - \lambda)^{-m-1}(z' - \lambda)^{-n-1} R_A(z') R_A(z) d\lambda' dz
\]

\[
= \left( \frac{1}{2\pi i} \right)^2 \oint_{\Gamma} \oint_{\Gamma'} (z - \lambda)^{-m-1}(z' - \lambda)^{-n-1} R_A(z) \frac{R_A(z')}{z' - z} d\lambda' dz
\]

\[
= \left( \frac{1}{2\pi i} \right)^2 \oint_{\Gamma} (z - \lambda)^{-m-1} R(z) \left[ \oint_{\Gamma'} (z' - \lambda)^{-n-1}(z' - z)^{-1} d\lambda' \right] dz
\]

\[
- \left( \frac{1}{2\pi i} \right)^2 \oint_{\Gamma'} (z' - \lambda)^{-n-1} R(z') \left[ \oint_{\Gamma} (z - \lambda)^{-m-1}(z' - z)^{-1} d\lambda \right] d\lambda'
\]

\[
= \frac{1}{2\pi i} \oint_{\Gamma} (z - \lambda)^{-m-1} R(z)(1 - \eta_n)(z - \lambda)^{-n-1} dz
\]

\[
- \frac{1}{2\pi i} \oint_{\Gamma'} (z' - \lambda)^{-n-1} R(z') \eta_m(z' - \lambda)^{-m-1} d\lambda'
\]
The integrand of the second integral has an isolated singularity at \( \lambda \), and so by the Cauchy-Goursat Theorem and the appropriate cut we can replace \( \Gamma' \) with \( \Gamma \) and \( z' \) with \( z \) without changing the the value of the second integral.

This gives for \( A_nA_m \) the expression

\[
\frac{1}{2\pi i} \oint_{\Gamma} [(z - \lambda)^{m-1}R(z)(1 - \eta_n)(z - \lambda)^{-n-1} - (z - \lambda)^{-n-1}R(z)\eta_m(z - \lambda)^{m-1}] \, dz
\]

\[
= \frac{1}{2\pi i} \oint_{\Gamma} (z - \lambda)^{m-1}(z - \lambda)^{-n-1}R(z)[1 - \eta_n - \eta_m] \, dz
\]

\[
= [1 - \eta_m - \eta_n] \frac{1}{2\pi i} \oint_{\Gamma} (z - \lambda)^{m-n-2}R(z) \, dz
\]

\[
= [1 - \eta_m - \eta_n] A_{m+n+1}.
\]

This gives the identity \( A_nA_m = (1 - \eta_m - \eta_n)A_{m+n+1} \). □

Remark 12.5.4. Since \( P_{\lambda} = A_{-1} \), Lemma 12.5.3 gives another proof that

\[
P^2_{\lambda} = A_{-1}A_{-1} = (1 - \eta_{-1} - \eta_{-1})A_{-1-1+1} = A_{-1} = P_{\lambda}.
\]

Notation. To express the relationships that exists among the coefficient matrices \( A_k \) in the Laurent series of \( R_A(z) \) about \( \lambda \), we define

\[
D_{\lambda} = A_{-2} \text{ and } S_{\lambda} = A_0.
\]

Lemma 12.5.5. For \( A \in M_n(\mathbb{C}) \) and \( \lambda \in \sigma(A) \), there holds

(i) \( A_n = D_{\lambda}^{n-1} \) for all \( n \geq 2 \),
(ii) \( A_n = (-1)^nS_{\lambda}^{n+1} \) for all \( n \geq 1 \),
(iii) the spectral projection \( P_{\lambda} \) commutes with \( D_{\lambda} \) and with \( S_{\lambda} \), where in particular,

\[
P_{\lambda}D_{\lambda} = D_{\lambda}, \ P_{\lambda}S_{\lambda} = 0,
\]

(iv) The Laurent series of \( R_A(z) \) about \( \lambda \) is

\[
R_A(z) = \frac{P_{\lambda}}{z - \lambda} + \sum_{k=1}^{\infty} \frac{D_{\lambda}^k}{(z - \lambda)^{k+1}} + \sum_{k=0}^{\infty} (-1)^k(z - \lambda)^kS_{\lambda}^{k+1},
\]

(v) the spectral projection \( P_{\lambda} \) commutes with \( R_A(z) \), where in particular

\[
P_{\lambda}R_A(z) = \frac{P_{\lambda}}{z - \lambda} + \sum_{k=1}^{\infty} \frac{D_{\lambda}^k}{(z - \lambda)^{k+1}}.
\]

The proof of these is HW (Exercises 12.23, 12.24, and 12.25).

Remark. The Laurent series for \( R_A(z) \) about \( \lambda \in \sigma(A) \) is completely determined by three matrices \( P_{\lambda} = A_{-1}, D_{\lambda} = A_{-2}, \) and \( S_{\lambda} = A_0 \).
Example. We verify some parts of Lemma 12.5.5 for the linear operator

\[ A = \begin{bmatrix} 6 & 1 & 0 \\ 6 & 1 & 7 \\ 0 & 0 & 4 \end{bmatrix} \]

and use other parts of Lemma 12.5.5 to compute Laurent series expansion of \( R_A(z) \) about \( \lambda = 6 \).

We computed previously that

\[ R_A(z) = \frac{1}{z - 6} \begin{bmatrix} 1 & 0 & -7/4 \\ 0 & 1 & 7/2 \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{(z - 6)^2} \begin{bmatrix} 0 & 1 & 7/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{z - 4} \begin{bmatrix} 0 & 0 & 7/4 \\ 0 & 0 & -7/2 \\ 0 & 0 & 1 \end{bmatrix}, \]

so that the spectral projections are

\[ P_6 = \begin{bmatrix} 1 & 0 & -7/4 \\ 0 & 1 & 7/2 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad P_4 = \begin{bmatrix} 0 & 0 & 7/4 \\ 0 & 0 & -7/2 \\ 0 & 0 & 1 \end{bmatrix}. \]

Also from the partial fraction decomposition of \( R_A(z) \) we have

\[ D_6 = \begin{bmatrix} 0 & 1 & 7/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad D_4 = 0. \]

We may thus neatly write

\[ R_A(z) = \frac{P_6}{z - 6} + \frac{D_6}{(z - 6)^2} + \frac{P_4}{z - 4}. \]

Verifying part (iii) of Lemma 12.5.5, the matrices \( P_6 \) and \( D_6 \) satisfy

\[ P_6 D_6 = \begin{bmatrix} 1 & 0 & -7/4 \\ 0 & 1 & 7/2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 7/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = D_6 \]

\[ = \begin{bmatrix} 0 & 1 & 7/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = D_6 \]

\[ = \begin{bmatrix} 0 & 1 & 7/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & -7/4 \\ 0 & 1 & 7/2 \\ 0 & 0 & 0 \end{bmatrix} = D_6 P_6. \]

The matrix \( D_6 \) satisfies

\[ D_6^2 = \begin{bmatrix} 0 & 1 & 7/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 7/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0. \]
Thus $D_6^k = 0$ for all $k \geq 2$, so that
\[
\sum_{k=1}^{\infty} \frac{D_6^k}{(z-6)^{k+1}} = \frac{D_6}{(z-6)^2}.
\]
We could compute $S_6$ by writing $1/(z-4)$ as a geometric series in $(z-6)$.
Instead we make use of parts (iv) and (v) of Lemma 12.5.5. First, by part (iv) we have
\[
\sum_{k=0}^{\infty} (-1)^k(z-6)^k S_6^{k+1} = R_A(z) - \left( \frac{P_6}{z-6} + \frac{D_6}{(z-6)^2} \right).
\]
By part (v) we have
\[
R_A(z) P_6 = \frac{P_6}{z-6} + \frac{D_6}{(z-6)^2}.
\]
Combining these gives
\[
\sum_{k=0}^{\infty} (-1)^k(z-6)^k S_6^{k+1} = R_A(z) - R_A(z) P_6 = R_A(z)(I - P_6) = R_A(z) P_4
\]
by the completeness $P_6 + P_4 = I$.
In the product $R_A(z) P_4$ we have $P_6 P_4 = 0$ and $P_4^2 = 0$, but what is $D_6 P_4$? It is
\[
\begin{bmatrix}
0 & 1 & 7/2 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 7/4 \\
0 & 0 & -7/2 \\
0 & 0 & 1
\end{bmatrix} = 0.
\]
Is this just a coincidence? According to part (v) of Lemma 12.5.5 it is not!
We thus have
\[
R_A(z) P_4 = \frac{P_4}{z-4}.
\]
The point of all of this is that we have
\[
\sum_{k=0}^{\infty} (-1)^k(z-6)^k S_6^{k+1} = \frac{P_4}{z-4}.
\]
Evaluating this equality at $z = 6$ gives
\[
S_6 = \frac{P_4}{2}.
\]
Since $P_6 P_4 = 0$, we verify part (iii) of Lemma 12.5.5 in that $P_6 S_6 = S_6 P_6 = 0$.
Since $P_4^2 = P_4$, we obtain $S_6^{k+1} = (1/2)^{k+1} P_4$, thus giving the Laurent series of the resolvent about $\lambda = 6$, namely
\[
R_A(z) = \frac{D_6}{(z-6)^2} + \frac{P_6}{z-6} + P_4 \sum_{k=0}^{\infty} \frac{(-1)^k(z-6)^k}{2^{k+1}}.
\]
Using the geometric series one can verify that

\[
\sum_{k=0}^{\infty} \frac{(-1)^k(z - 6)^k}{2^{k+1}} = \frac{1}{z - 4}.
\]

Example (in lieu of 12.5.6). We compute the Laurent series

\[
R_A(z) = \sum_{k=-\infty}^{\infty} A_k(z - 2)^k
\]

for the linear operator

\[
A = \begin{bmatrix}
2 & 1 & 0 & 0 \\
0 & 2 & 1 & 0 \\
0 & 0 & 2 & 3 \\
0 & 0 & 0 & 5
\end{bmatrix}.
\]

To this end we need \(P_2, D_2,\) and \(S.\)

We computed previously that

\[
R_A(z) = \frac{1}{z - 2} \begin{bmatrix}
1 & 0 & 0 & -1/9 \\
0 & 1 & 0 & -1/3 \\
0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0
\end{bmatrix} + \frac{1}{(z - 2)^2} \begin{bmatrix}
0 & 1 & 0 & -1/3 \\
0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
+ \frac{1}{(z - 2)^3} \begin{bmatrix}
0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} + \frac{1}{z - 5} \begin{bmatrix}
0 & 0 & 0 & 1/9 \\
0 & 0 & 0 & 1/3 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1
\end{bmatrix}.
\]

The spectral projections are

\[
P_2 = \begin{bmatrix}
1 & 0 & 0 & -1/9 \\
0 & 1 & 0 & -1/3 \\
0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0
\end{bmatrix} \quad \text{and} \quad P_5 = \begin{bmatrix}
0 & 0 & 0 & 1/9 \\
0 & 0 & 0 & 1/3 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1
\end{bmatrix}.
\]

We also have

\[
D_2 = A_{-2} = \begin{bmatrix}
0 & 1 & 0 & -1/3 \\
0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

and by way of verification that

\[
D_2^2 = \begin{bmatrix}
0 & 1 & 0 & -1/3 \\
0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
0 & 1 & 0 & -1/3 \\
0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} = A_{-3}.
\]
To find $S_2$ we have by part (iv) of Lemma 12.5.5 that
\[
\sum_{k=0}^{\infty} (-1)^k (z - 2)^k S_2^{k+1} = R_A(z) - \left( \frac{P_2}{z - 2} + \frac{D_2}{(z - 2)^2} \right)
\]
and by part (v) of Lemma 12.5.5 that
\[
P_2 R_A(z) = \frac{P_2}{z - 2} + \frac{D_2}{(z - 2)^2}.
\]
Combining these gives
\[
\sum_{k=0}^{\infty} (-1)^k (z - 2)^k S_2^{k+1} = R_A(z) - P_2 R_A(z) = (I - P_2) R_A(z) = R_A(z) P_5 = \frac{P_5}{z - 5},
\]
where we have used the completeness $P_2 + P_5 = I$ and part (iv) of Lemma 12.5.5 applied to $\lambda = 5$.

Evaluation of the equality at $z = 2$ gives $S_2 = -(1/3)P_5$.

[Note the book incorrectly says to integrate to get this for the example it considers.]

Thus the Laurent series for the resolvent of $A$ around $\lambda = 2$ is
\[
R_A(z) = \frac{D_2^2}{(z - 2)^3} + \frac{D_2}{(z - 2)^2} + \frac{P_2}{z - 2} - P_5 \sum_{k=0}^{\infty} \frac{(z - 2)^k}{3^{k+1}}.
\]

Using the geometric series we can verify that
\[
- \sum_{k=0}^{\infty} \frac{(z - 2)^k}{3^{k+1}} = \frac{1}{z - 5}.
\]

We mentioned previously that $A \neq 2P_2 + 5P_5$ since $A$ is not semisimple, but that something else was happening.

The spectral decomposition of $A$ is $2P_2 + D_2 + 5P_5$ because
\[
2P_2 + D_2 + 5P_5 = 2 \begin{bmatrix} 1 & 0 & 0 & -1/9 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + 5 \begin{bmatrix} 0 & 0 & 0 & 1/9 \\ 0 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = A.
\]