§2.4: Bifurcations

There are many kinds of bifurcations that can occur in a parameter dependent system of nonlinear equations.

We will give examples of two of them.

Example (Saddle-Node Bifurcation). Consider the nonlinear system

\[ x' = \mu - x^2, \quad y' = -y. \]

Each equation can be solved explicitly, but we will use the geometric analysis approach to see how the solutions change as \( \mu \) is varied.

The phase line for \( y' = -y \) shows that \( y(t) \to 0 \) for all choices of initial \( y \) value.

For \( \mu < 0 \) we have that \( x' < -\mu \), so there are no equilibria, and \( x(t) \) decreases with bound.

For \( \mu = 0 \) we have one equilibrium at the origin with \( x' > 0 \) for \( x < 0 \) and \( x > 0 \) (so the equilibrium is unstable).

The Jacobian for the linearization at the origin (when \( \mu = 0 \)) is diagonal with entries 0 and \(-1\), which are the eigenvalues, and so the equilibrium is non-hyperbolic.
For $\mu > 0$ we have two equilibria $(x^*, 0)$ for $x^* = \pm \sqrt{\mu}$.

The Jacobian of the linearization at the equilibrium $(-\sqrt{\mu}, 0)$ is diagonal with entries $2\sqrt{\mu}$ and $-1$, and so it is a saddle point.

The Jacobian of the linearization at the equilibrium $(\sqrt{\mu}, 0)$ is diagonal with entries $-2\sqrt{\mu}$ and $-1$, and so it is a locally asymptotically stable node.

Hence at $\mu = 0$ we have a Saddle-Node Bifurcation.

Example (Hopf Bifurcation). Consider the nonlinear system

$$
\begin{align*}
x' &= -y + x(\mu - x^2 - y^2), \\
y' &= x + y(\mu - x^2 - y^2).
\end{align*}
$$

The only equilibrium is at the origin.

The Jacobian matrix is

$$
A = \begin{bmatrix} \mu & -1 \\ 1 & \mu \end{bmatrix}.
$$

The characteristic equation of $A$ is

$$
\lambda^2 - 2\mu + (\mu^2 + 1) = 0.
$$

The eigenvalues are

$$
\lambda = \frac{2\mu \pm \sqrt{4\mu^2 - 4(\mu^2 + 1)}}{2} = \mu \pm i.
$$

For $\mu > 0$ the equilibrium is an unstable spiral point,

For $\mu = 0$ the equilibrium is a linear center.

For $\mu < 0$ the equilibrium is a locally asymptotically stable spiral point.

What does the phase portrait look like when $\mu = 0$? And, what happens to the phase portrait as $\mu$ passes through $\mu = 0$ from negative to positive?

These can be explicitly answered by transforming the equations into polar coordinates.
With \( x = r \cos \theta \), \( y = r \sin \theta \) we have
\[
xx' + yy' = r \cos \theta (r' \cos \theta - r \theta' \sin \theta) + r \sin \theta (r' \sin \theta + r \theta' \cos \theta)
\]
\[
= rr' \cos^2 \theta - r^2 \theta^2 \cos \theta \sin \theta + RR' \sin^2 \theta + r^2 \sin \theta \cos \theta
\]
\[
= rr'
\]
and
\[
xy' - yx' = r \cos \theta (r' \sin \theta + r \theta' \cos \theta) - r \sin \theta (r' \cos \theta - r \theta' \sin \theta)
\]
\[
= rr' \cos \theta \sin \theta + r^2 \theta' \cos^2 \theta - rr' \sin \theta \cos \theta + r^2 \theta' \sin^2 \theta
\]
\[
= r^2 \theta'.
\]
Using the nonlinear system we have
\[
rr' = xx' + yy' = -xy + x^2(\mu - x^2 - y^2) + xy + y^2(\mu - x^2 - y^2) = r^2(\mu - r^2)
\]
and
\[
r^2 \theta' = xy' - yx' = x^2 + xy(\mu - x^2 - y^2) + y^2 - xy(\mu - x^2 - y^2).
\]
The nonlinear system in polar coordinates is
\[
r' = r(\mu - r^2), \quad \theta' = 1.
\]
To understand the nonlinear system, we apply the geometric analysis approach to \( r' = r(\mu - r^2) \) (although we could solve this separable equation as the book does).

For \( \mu < 0 \) we have that \( r' < 0 \) for all \( r > 0 \), so that all non-equilibrium solutions in the \( xy \)-plane tend to the origin (which in the linearization is locally asymptotically stable equilibrium).

For \( \mu = 0 \), we have that \( r' < 0 \) for all \( r > 0 \), so that all non-equilibrium solutions in the \( xy \)-plane tend to the origin, and so the origin is a not a center (although the linearization is a center).
For each fixed $\mu > 0$, the constant function $r(t) = \sqrt{\mu} > 0$ is a solution of $r' = r(\mu - r^2)$.
Coupled with $\theta' = 1$, we get a periodic solution $r(t) = \sqrt{\mu}$, $\theta(t) = t + \theta_0$ in the $xy$-plane.

The sign of $r' = r(\mu - r^2)$ when $0 < r < \sqrt{\mu}$ is positive, so solutions starting near the origin in the $xy$-plane spiral away from it (the origin is an unstable spiral point), and move towards the periodic orbit $r = \sqrt{\mu}$.

The sign of $r' = r(\mu - r^2)$ when $r > \sqrt{\mu}$ is negative, so solutions starting with large $r$ value in the $xy$-plane spiral towards the period orbit $r = \sqrt{\mu}$.

The periodic orbit $r = \sqrt{\mu}$ is called a **stable limit cycle**.

The appearance of a stable (unstable) limit cycle from a stable (unstable) equilibrium as the parameter varies is known as a Hopf bifurcation.