Recall that last time we derive a second-order differential equation that a local minimizer must satisfy for the arc length functional.

The derivation can be carried out for functionals of the form

\[ J(y) = \int_{a}^{b} L(x, y, y') \, dx \]

where the Lagrangian \( L(x, y, y') \) is a twice continuously differentiable function defined on \([a, b] \times \mathbb{R} \times \mathbb{R}\).

A key step is the following result.

**Lemma 4.13.** If \( f(x) \) is continuous on \([a, b]\) and if

\[ \int_{a}^{b} f(x) h(x) \, dx = 0 \]

for every \( h \in C^2[a, b] \) with \( h(a) = 0 \) and \( h(b) = 0 \), then \( f(x) = 0 \) for all \( x \in [a, b] \).

**Proof.** Assume, by way of contradiction, that there is \( x_0 \in [a, b] \) such that \( f(x_0) \neq 0 \).

With loss of generality, we may assume that \( f(x_0) > 0 \).

Because \( f \) is continuous, there is an interval \([x_1, x_2]\) with \( a \leq x_1 < x_2 \leq b \) such that \( x_0 \in [x_1, x_2] \).

For a function \( h(x) \in C^2[a, b] \) with \( h(a) = 0 \) and \( h(b) = 0 \), we choose

\[ h(x) = \begin{cases} (x - x_1)^3(x_2 - x)^3 & \text{if } x_1 \leq x \leq x_2, \\ 0 & \text{otherwise.} \end{cases} \]

Here is the graph of this function \( h(x) \).
The reason for the cubic powers is to ensure that the first and second derivatives of $h(x)$ at $x_1$ and $x_2$ are all zero, thus ensuring that $h \in C^2[a, b]$.

With this choice of $h(x)$ we reach the contradiction,

$$\int_a^b f(x)h(x) \, dx = \int_{x_1}^{x_2} f(x)(x - x_1)^3(x_2 - x)^3 \, dx > 0.$$ 

This shows that $f(x) = 0$ for all $x \in [a, b]$. \qed

We can now state the necessary condition of a differential equation that a local minimizer must satisfy.

The differential equations obtained do not depend on the choice of a norm on the function space which contains the set of admissible functions.

**Theorem 4.14.** If $y_0 \in A = \{y \in C^2[a, b] : y(a) = y_a, \ y(b) = y_b\}$ is a local minimizer of the functional

$$J(y) = \int_a^b L(x, y, y') \, dx,$$

then $y_0$ must satisfy the **Euler equation** (or **Euler-Lagrange equation**),

$$L_y(x, y, y') - \frac{d}{dx}L_{y'}(x, y, y') = 0.$$

**Proof.** For $h \in C^2[a, b]$ with $h(a) = 0$ and $h(b) = 0$, the variation $y_0 + \epsilon h$ is admissible for small enough $\epsilon$.

Then

$$J(y_0 + \epsilon h) = \int_a^b L(x, y + \epsilon h, y' + \epsilon h') \, dx,$$

and so

$$\frac{d}{d\epsilon} J(y_0 + \epsilon h) = \int_a^b \frac{\partial}{\partial \epsilon} L(x, y_0 + \epsilon h, y'_0 + \epsilon h') \, dx$$

$$= \int_a^b \left[ L_y(x, y_0 + \epsilon h, y'_0 + \epsilon h')h + L_{y'}(x, y_0 + \epsilon h, y'_0 + \epsilon h')h' \right] \, dx,$$

where $L_y = \partial L/\partial y$ and $L_{y'} = \partial L/\partial y'$.

Thus

$$\left. \frac{d}{d\epsilon} J(y_0 + \epsilon h) \right|_{\epsilon=0} = \int_a^b \{ L_y(x, y_0, y'_0)h + L_{y'}(x, y_0, y'_0)h' \} \, dx.$$

Since $y_0$ is a local minimizer, we know that

$$\delta J(y_0, h) = \frac{d}{d\epsilon} J(y_0 + \epsilon h) = 0$$

for all $h \in C^2[a, b]$ with $h(a) = 0$ and $h(b) = 0$. 

This implies that
\[ \int_a^b \left\{ L_y(x, y_0, y'_0)h + L_{y'}(x, y_0, y'_0)h' \right\} dx = 0 \]
holds for all \( h \in C^2[a, b] \) with \( h(a) = 0 \) and \( h(b) = 0 \).

We perform integration by parts on the second term in the integral: with \( u = L_{y'}(x, y_0, y'_0) \) and \( dv = h' dx \), we get
\[ \int_a^b \left( L_y(x, y_0, y'_0) - \frac{d}{dx} L_{y'}(x, y_0, y'_0) \right) h \ dx + L_{y'}(x, y_0, y'_0)h \bigg|_{x=a}^{x=b} = 0. \]

Because \( h(a) = 0 \) and \( h(b) = 0 \), the evaluations vanishes, and we obtain
\[ \int_a^b \left( L_y(x, y_0, y'_0) - \frac{d}{dx} L_{y'}(x, y_0, y'_0) \right) h \ dx = 0 \]
which holds for all \( h \in C^2[a, b] \) with \( h(a) = 0 \) and \( h(b) = 0 \).

We apply Lemma 4.13 to obtain
\[ L_y(x, y_0, y'_0) - \frac{d}{dx} L_{y'}(x, y_0, y'_0) = 0, \]
which is the Euler equation. \( \square \)

This necessary condition was derived from the assumption of a local minimizer.

We do NOT know if any solution \( y \) of the Euler equation will be a local minimizer of \( J(y) \).

However, any solution of the Euler equation \( y \) will satisfy \( \delta J(y, h) = 0 \) for all \( h \) (we get this by working some of the steps of the proof of the Theorem backwards).

So, in general, each solution of the Euler equation is a local extremum of \( J \).

Carrying out the derivative with respect to \( x \) in the Euler equation gives
\[ L_y(x, y, y') + L_{y'y'}(x, y, y') + L_{y'y''}(x, y, y')y'' + L_{y'y'''}(x, y, y')y''' = 0 \]
and so the Euler equation is a second order differential equation when \( L_{y'y'} \neq 0 \).

Example. The Euler equation for
\[ J(y) = \int_0^1 \left( \frac{m(y')^2}{2} - V(y) \right) \ dx \]
is
\[ 0 = L_y(x, y, y') - \frac{d}{dx} L_{y'}(x, y, y') = -V'(y) - my''. \]

The extremals of \( J \) are precisely the solutions of the mechanical system \( my'' = -V'(y) \).