4.1: The Lebesgue outer measure in $\mathbb{R}^N$. Let $Q$ denote the collection of all $\frac{1}{2}$-closed dyadic cubes in $\mathbb{R}^N$.

These cubes are of the form

$$Q_{p,q} = \left\{ x \in \mathbb{R}^N : \frac{q_i - 1}{2^p} < x_i \leq \frac{q_i}{2^p} \right\}.$$

For such a cube $Q$, we denote by $\text{diam}(Q)$ the length of a diameter of the cube:

$$\text{diam}(Q) = \sup_{x,y \in Q} |x - y| = \sqrt{\left( \frac{1}{2^p} \right)^2 + \cdots + \left( \frac{1}{2^p} \right)^2} = \frac{\sqrt{N}}{2^p}.$$ 

For the function $\lambda$ defined on $Q$ we take

$$\lambda(Q) = \left( \frac{\text{diam}(Q)}{\sqrt{N}} \right)^N = \left( \frac{\sqrt{N}}{2^p \sqrt{N}} \right)^N = \left( \frac{1}{2^p} \right)^N,$$

which is the volume of $Q$.

Since every subset of $\mathbb{R}^N$ can be contained in some open subset, and since $Q$ is a sequential covering of $\mathbb{R}^N$, we can define for every $E \in \mathcal{P}(\mathbb{R}^N)$,

$$\mu_e(E) = \inf \left\{ \sum \left( \frac{\text{diam}(Q_n)}{\sqrt{N}} \right)^N : E \subset \bigcup Q_n, \{Q_n\} \subset Q \right\},$$

which is the Lebesgue outer measure of $\mathbb{R}^N$.

4.2: The Lebesgue-Stieltjes outer measure in $\mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}$ be a monotone increasing and right-continuous, i.e., $x \leq y$ implies $f(x) \leq f(y)$, and $\lim_{x \to a^+} f(x) = f(a)$ for all $a \in \mathbb{R}$ (where the right-sided limit exists and is equal to $\inf_{x>a} f(x)$ by the monotone increasing of $f$).

Let $Q$ be the collection of open subintervals of $\mathbb{R}$, which is a sequential covering of $\mathbb{R}$ (see Proposition 1 in II.1).

Define $\lambda_f : Q \to \mathbb{R}$ by

$$\lambda_f((a,b)) = f(b) - f(a).$$

For $E \in \mathcal{P}(\mathbb{R})$, the outer measure generated by $Q$ and $\lambda$ is

$$\mu_{f,e}(E) = \inf \left\{ \sum \lambda_f(Q_n) : E \subset \bigcup Q_n, \{Q_n\} \subset Q \right\},$$

and is called the Lebesgue-Stieltjes outer measure generated by $f$. 
Proposition. $\mu_{f,e}((a, b]) = f(b) - f(a)$.

Proof. Recall for a nonempty bounded below set of real numbers $S$ and a lower bound $\alpha$ of $S$, that $\alpha = \inf S$ if and only if for all $\epsilon > 0$ there exists $s \in S$ such that $s < \alpha + \epsilon$. Thus, if we can show that $f(b) - f(a)$ is a lower bound, and for every $\epsilon > 0$ we can exhibit the existence of a sequential covering $\{Q_n\}$ of $(a, b]$ such that

$$\sum \lambda_f(Q_n) < f(b) - f(a) + \epsilon,$$

then we have implied that $\mu_{f,e}((a, b]) = f(b) - f(a)$.

To show that $f(b) - f(a)$ is a lower bound, suppose that it is not. Then there is $\{Q_n\} \subset Q$ such that $(a, b] \subset \bigcup Q_n$ and

$$\sum \lambda_f(Q_n) < f(b) - f(a).$$

Since $(a, b] \subset \bigcup Q_n$, there is an open interval $(c, d)$ such that $(a, b] \subset (c, d) \subset \bigcup Q_n$.

Since $(c, d) \subset \bigcup Q_n = \bigcup (a_n, b_n)$, the monotonicity of $f$ and our assumption imply

$$f(d) - f(c) = \lambda_f((c, d)) \leq \sum \lambda_f((a_n, b_n)) = \sum \lambda_f(Q_n) < f(b) - f(a).$$

But $c \leq a < b < d$ which implies that $f(b) - f(a) \leq f(d) - f(c)$, a contradiction.

For $\epsilon > 0$ there is by the right continuity of $f$ at $b$ the existence of $\delta > 0$ such that for all $x \in (b, b + \delta)$ we have $f(x) - f(b) < \epsilon$.

Select $b_\epsilon \in (b, b + \delta)$.

The sequential covering of $(a, b]$ consisting of just the one open interval $(a, b_\epsilon)$ satisfies

$$\lambda_f((a, b_\epsilon)) = f(b_\epsilon) - f(a) < f(b) - f(a) + \epsilon.$$

Therefore we have that $\mu_{f,e}((a, b]) = f(b) - f(a)$.

Does $\mu_{f,e}((a, b]) = \lambda_f((a, b])$ for all $(a, b) \in Q$?

Proposition. If $f$ is discontinuous at $b$ but continuous on $[a - \eta, b)$ for a small $\eta > 0$, then $\mu_{f,e}((a, b)) < f(b) - f(a)$.

Proof. Suppose $f$ is discontinuous at $b$, i.e., there is a jump discontinuity of $f$ at $b$, while $f$ is continuous on $[a - \eta, b)$ for some small $\eta > 0$.

Set

$$\epsilon = \lim_{x \to b^-} f(x) - \lim_{x \to b^+} f(x) = f(b) - \lim_{x \to b^-} f(x) > 0.$$

We will choose a sequential covering $\{Q_n\}$ of $(a, b)$ as follows.

We want the endpoints of the covering intervals $Q_n = (a_n, b_n)$ to satisfy

1. $\{a_n\}$ is strictly increasing with $a_1 = a$,
2. $\{b_n\}$ is strictly increasing with $b_n < b$ for all $n$,
3. $a_{n+1} < b_n$ for all $n$. 


Since $b_n \in [a - \eta, b)$, we can choose $a_{n+1}$ close enough to $b_n$ by the continuity of $f$ at $b_n$ so that
\[ f(b_n) - f(a_{n+1}) \leq \frac{\epsilon}{2^{n+1}}. \]
Then we have for each positive integer $m$ that
\[
\sum_{n=1}^{m} (f(b_n) - f(a_n)) = -f(a_1) + (f(b_1) - f(a_2)) + (f(b_2) - f(a_3)) + \cdots \\
+ (f(b_{m-1}) - f(a_m)) + f(b_m) \\
= f(b_m) - f(a_1) + \sum_{n=1}^{m-1} \frac{\epsilon}{2^{n+1}} \leq \lim_{x \to b^-} f(x) - f(a) + \frac{\epsilon}{2},
\]
where we have used the monotonicity of $f$.
This upper bound on the partial sums implies that
\[
\sum_{n=1}^{\infty} \lambda_f(Q_n) < \lim_{x \to b^-} f(x) - f(a) + \epsilon \\
= \lim_{x \to b^-} f(x) - f(a) + f(b) - \lim_{x \to b^-} f(x) \\
= f(b) - f(a).
\]
Therefore $\mu_{f,\epsilon}((a, b)) < f(b) - f(a) = \lambda_f((a, b))$. \hfill \Box

**Homework Problem 7A.** For $f(x) = e^x$, find $\mu_{f,\epsilon}((a, b))$.

5. **Hausdorff Outer Measure in** $\mathbb{R}^N$. For $\epsilon > 0$, let $\mathcal{E}_\epsilon$ be the sequential covering of $\mathbb{R}^N$ consisting of all subsets $E$ of $\mathbb{R}^N$ for which $\text{diam}(E) < \epsilon$.

For fixed $\alpha > 0$ and $E \in \mathcal{E}_\epsilon$, define $\lambda(E) = (\text{diam}(E))^{\alpha}$, and set $\lambda(\emptyset) = 0$.

The nonnegative function $\lambda$ on $\mathcal{E}_\epsilon$ generates an outer measure
\[
\mathcal{H}_{\alpha,\epsilon}(E) = \inf \left\{ \sum_{i=1}^{\infty} (\text{diam}(E_n))^\alpha : E \subset \bigcup_{n} E_n, \{E_n\} \in \mathcal{E}_\epsilon \right\}.
\]
For this outer measure, we have $\mathcal{H}_{\alpha,\epsilon}(E) < \lambda(E)$ when $\alpha > 2$ and $E$ is a square of unit edge in $\mathbb{R}^2$ because
\[
\lambda(E) = (\sqrt{1 + 1})^\alpha = (\sqrt{2})^\alpha,
\]
while for all $\alpha > 2$ there holds
\[
\mathcal{H}_{\alpha,\epsilon}(E) = 0.
\]

**Homework problem 7B.** Prove that $\mathcal{H}_{\alpha,\epsilon}(E) = 0$ for all $\alpha > 2$.

If $\epsilon' < \epsilon$ then $\mathcal{H}_{\alpha,\epsilon} \leq \mathcal{H}_{\alpha,\epsilon'}$ because every sequential covering by sets in $\mathcal{E}_{\epsilon'}$ is a sequential covering by sets in $\mathcal{E}_\epsilon$, but not every sequential covering by sets in $\mathcal{E}_{\epsilon'}$ is a sequential covering by sets in $\mathcal{E}_\epsilon$. This implies that
\[
\mathcal{H}_\alpha(E) = \sup_{\epsilon > 0} \mathcal{H}_{\alpha,\epsilon}(E) = \lim_{\epsilon \to 0} \mathcal{H}_{\alpha,\epsilon}(E)
\]
exists in $\mathbb{R}^*$ and defines a nonnegative function $\mathcal{H}_\alpha$ on $\mathcal{P}(\mathbb{R}^N)$. 